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New Perspectives on the Laminar Boundary Layer Physics in a Polarized Pressure Field with Temperature Gradient: an Analytical Approximation to Blasius Equation

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ABSTRACT

This study proposes a semi-analytic approximation to the laminar boundary layer growth in a polarized pressure field with temperature gradient represented by the joint Blasius-energy equation. We illuminate that $f''(\eta)$ is a probability density function (PDF) approximated by an amended Gaussian PDF with zero mean and standard deviation $\sigma = 2.18$. This implies a diffusive structure for the molecular momentum conversion as well as the energy flux in the boundary layer. A new limit for the boundary layer edge is also presented. Results suggest an augmented boundary layer when compared to accepted values in the literature. We also reproduce the inverse proportionality of the free stream velocity to the diffusion of both momentum and energy.

Keywords: Blasius laminar flow; Semi-analytic approximation; Boundary layer thickness; Momentum diffusion; Energy diffusion.

NOMENCLATURE

a	constant determined to be = 9.24sqrt(\nu / U0)
c_p	fluid specific heat
c_0	$= \mathrm{erf}\left[a\sqrt{U_0/(2\upsilon\sigma^2)}\right]$
D_b	uniform-pressure momentum constant
D_t	uniform-pressure thermal constant
k	thermal conductivity
k^*	mean absorption coefficient
k_0	$= 3N_R/(3N_R+4)$
N_R	$= k.k^*/4\sigma^* T_\infty^3$ Radiation parameter
p	piezometric pressure
$\breve{Pr}$	modified Prandtl number
Pr	Prandtl number
U_0	free stream velocity
T	fluid temperature
T_∞	temperature of the ambient fluid
T_w	temperature of the wall
u	streamwise velocity
v	crosswise velocity
ρ	density of fluid
$f'(\eta)$	$= u/U_0$
α	thermal diffusivity
υ	kinematic viscosity
θ	dimensionless temperature defined in Eq. (6)
σ^*	Stefan–Boltzmann constant
η	$= y\sqrt{U_0/\upsilon x}$
$\hat{\phi}(\mu,\sigma)$	amended Gaussian PDF defined in Eq. (9)
μ	=0 (Mean of the Gaussian distribution)
$\phi(\mu,\sigma)$	Gaussian PDF with mean μ and standard deviation σ
σ	=2.18 (Standard deviation of the Gaussian distribution)
δ	boundary layer thickness

1. INTRODUCTION

The thorough investigation of inviscid flows led to the increased awareness and development of the boundary layer concept (Schlichting, 1979). This notion, which was largely evolved by the work of Prandtl, introduced various levels of simplification of the governing equations of fluid flow (Hirsch, 2007). A direct consequence was the feasibility of simulation of industrial-scaled applications which used to be recalcitrant by the full Navier-Stokes equations (Chen *et al.*, 1984; Raptis *et al.*, 2004; Rashidi and Erfani, 2011; Lu and Law, 2014).

One of the most spectacular examples of boundary layers was studied by Blasius almost a century ago. It was based on the low-speed laminar flow passing over a flat plate in a polarized pressure field (Blasius,

1950). Along with experimental visualizations, Blasius exploited the Prandtl equations in conjunction with assumptions about the absence of pressure gradient and similarity of velocity profiles to elaborate the so-called Blasius equation. With the help of recursive power series, Blasius (1950) solved it with satisfying levels of accuracy. Subsequently, Howarth (1938) performed the Runge-Kutta method to provide an enhanced numerical scheme.

Due to the engineering significance and mathematical value of Blasius equation, there has appeared an extensive body of literature surrounding it. For instance, Healey (2008) investigated the effect of a controlled point-source disturbance in the Blasius laminar regime. By means of a wave-envelope steepening mechanics, they illustrated the extent to which the flow becomes nonlinear at low amplitudes. Kuo (2004) presented the solution to the momentum and energy equations of Blasius flow in the presence of temperature gradient. Zuccher *et al*. (2006) suggested a numerical scheme to solve the whole boundary layer equations with the aim of determining the maximum energy growth in a surging instable Blasius flow. Abdallah and Zeghmati (2011) examined the relative tendency to heat and mass transfer through the boundary layer when exposed to buoyancy gradients in the vicinity of a cylindrical surface. Their results verified the dominance of mass transfer over energy scattering when the ratio of Prandtl to Schmidt number witnessed a significant plummet.

On the mathematical side, major advances in numerical/analytical techniques have been made to the need for an enhanced quantification of fluid motion problems. Such spectrum is very broad ranging from the revelation of homotopy analysis (Liao, 1992; Liao, 1999; Bég *et al*., 2012; Malvandi *et al*., 2014; Hassan and Rashidi, 2014 and Makukula and Motsa, 2014), differential transforms (Rashidi *et al*., 2013 and Ganji *et al*., 2016) and Adomian decompositions (Adomian, 1994; Wang, 2004; Aski *et al*., 2014 and Akpan, 2015) to a combination of these and analogous techniques (e.g., coupled integral transform and functional analysis (Lari and Moeini, 2015) and the joined differential transform method with the Padè approximants (Rashidi *et al*., 2013 and Thiagarajan and Senthilkumar, 2013). An important outcome was investigation into more complex physical problems. For example, the boundary layer development was examined under the influence of magnetic response, with results showing the possibility of devaluating the skin friction in an orthogonal magnetic field configuration (Thiagarajan and Senthilkumar, 2013; and Vyas and Srivastavat, 2012). In addition, several works arrived at a simulation for the behavior of non-Newtonian and thixotropic fluids in the boundary layers (Sadeqi *et al*., 2011; Pan *et al*., 2016 and Ashraf *et al*., 2016) which have been unknown until quite recently.

As a sequel to this corpus, our goal here is to suggest a semi-analytic approximation analogous to results in Ahmad and Al-Barakati (2009), Yun (2010), Savas (2012) and Bataller (2008) with the aim of providing more detailed analysis of the laminar boundary layer physics in a polarized pressure field with temperature gradient. The reason behind the appeal to this approximation, which makes it preferable to the comparable works, lies in its simplicity; In fact, we will justify that through this closed-form approximation, the important properties of boundary layer thickness/diffusion can be investigated with little computational expense.

2. GOVERNING EQUATIONS

The Prandtl equation (Bird *et al.*, 2001, p. 135]

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\upsilon\frac{\partial^2 u}{\partial y^2} \quad (1)$$

lays the theoretical foundation to identify the external flow characteristics. Here, u is the streamwise velocity, v the crosswise velocity, p the piezometric pressure and υ the kinematic viscosity. In the case of a uniform pressure field through the boundary layer, Prandtl equation reduces to the Blasius equation (Bird *et al.* 2001, p. 138]

$$f(\eta)\,f''(\eta)+2f'''(\eta)=0 \qquad \left\langle \begin{array}{l} f'(\eta)=\dfrac{u}{U_0} \\ \eta=y\sqrt{\dfrac{U_0}{\upsilon x}} \end{array} \right\rangle \quad (2)$$

Considering the no-slip condition on the plate ($u = v = 0$ at $y = 0$) and an asymptotically-reached potential flow outside the boundary layer ($u = U_0$ at $y = \infty$), Eq. (2) will get subject to

$$f\big|_{\eta=0}=0 \qquad f'\big|_{\eta=0}=0 \qquad f'\big|_{\eta=\infty}=1 \quad (3)$$

In addition, thermal radiation in the Blasius regime is typically formulated as (Bataller, 2008)

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\alpha}{k_0}\frac{\partial^2 T}{\partial y^2} \quad (4)$$

where T is the temperature, $\alpha = k / \rho c_p$ the thermal diffusivity, k the thermal conductivity, c_p the fluid specific heat, $k_0 = 3N_R/(3N_R+4)$ where $N_R = k.k^* / 4\sigma^* T_\infty^3$ is the radiation parameter, k^* the mean absorption coefficient and σ^* the Stefan–Boltzmann constant. This PDE can be transformed to (Bataller, 2008)

$$\theta''(\eta) + \frac{\Pr k_0}{2} f(\eta)\theta'(\eta) = 0 \tag{5}$$

where Pr is the *Prandtl number* and θ the *dimensionless temprature* defined as

$$\theta = \frac{T - T_w}{T_\infty - T_w} \tag{6}$$

where T_w is the temperature of the wall and T_∞ the temperature of the ambient fluid. The boundary conditions for Eq. (5) are

$$\theta_{|\eta=0} = 0; \quad \theta_{|\eta=\infty} = 1 \tag{7}$$

resulting from the temperature equilibrium at the interface of solid-fluid boundary ($T = T_w$ at $y = 0$) and the prevailing uniform ambient temperature at the far-field.

3. SEMI-ANALYTICAL APPROXIMATION

The present approximation is based on the hypothesis that a dimensionless quantity taking values in the interval [0, 1] can be regarded as a cumulative distribution function (CDF). By comparing the CDF against numerical schemes, we propose a closed-form solution to the ODE in this section, and portray some of its outcomes in section (4).

On the grounds of the important property (Iacono and Boyd, 2015)

$$\int_{-\infty}^{+\infty} f''(\eta)d\eta = \int_{-\infty}^{0} f''(\eta)d\eta + f'(\infty) - f'(0) = 1 \tag{8}$$

$f''(\eta)$ can be thought of a probability density function (PDF) (Roussas, 2003, p. 34). Vias and Srivastava (2012) and Iacono and Boyd (2015) justified that $f''(\eta)$ attains an exponential decline when η linearly grows. This provides us with a criterion to the need for a proximate estimation of the PDF. For this purpose, we define an *Amended Gaussian* function as

$$\hat{\phi}(\mu,\sigma) = \begin{cases} 2\phi(\mu,\sigma) & \eta \geq 0 \\ 0 & \eta < 0 \end{cases} \tag{9}$$

where $\phi(\mu,\sigma) = e^{-(\eta-\mu)^2/(2\sigma^2)}/(\sqrt{2\pi}\,\sigma)$ is the PDF of a Gaussian variable with mean μ and standard deviation σ (Roussas, 2003). The function (9) is called *the Amended Gaussian* PDF since the negative branch of the Gaussian PDF is cut and the positive values are doubled to keep the integral $\int_{-\infty}^{+\infty} \hat{\phi}(\mu,\sigma) = 1$ (analogous with the integral in Eq. (8) and with the definition of a probability density function). As a result of a qualitative interpolation, it can be seen that the option $\mu = 0$ and $\sigma = 2.18$ results in a close consistency between the two PDFs (see Fig. (1)). Therefore,

$$f''_\sigma(\eta) = \hat{\phi}(0,\sigma) \tag{10}$$

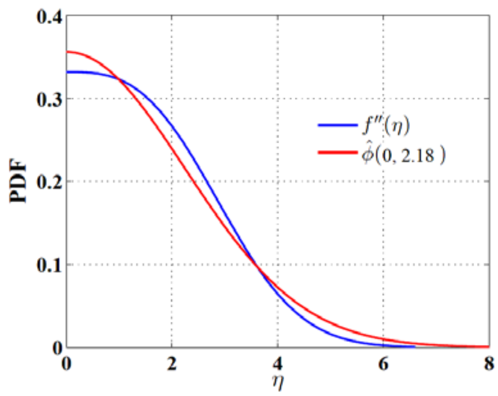

Fig. 1. Close consistency between the density function $f''_{\sigma=2.18}(\eta)$ and the Amended *Gaussian* PDF in Eq. (9).

This can be validated by solving ODE (10) to obtain the closed-form approximation

$$f_\sigma(\eta) = -\sigma\sqrt{\frac{2}{\pi}} + \sigma\sqrt{\frac{2}{\pi}}\, e^{-\left(\frac{\eta}{\sigma\sqrt{2}}\right)^2} + \eta \,\mathrm{erf}\left(\frac{\eta}{\sigma\sqrt{2}}\right) \tag{11}$$

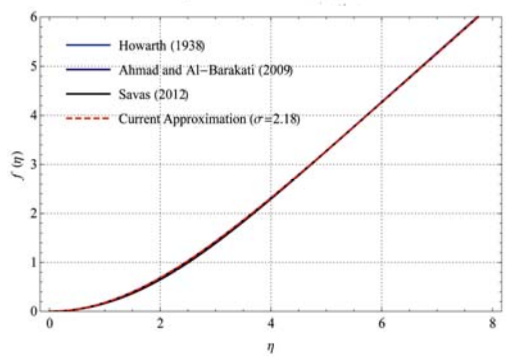

Fig. 2. Comparison of the suggested approximation $f_{\sigma=2.18}(\eta)$ with the comparable schemes in Howarth (1938), Ahmad and Al-Barakati (2009) and Savas (2012). Note that the results are coincident.

which is fully concordant with the exact solution of Blasius equation (e,g., Howarth, 1938 (numerical), Ahmad and Al-Barakati, 2009 and Savas, 2012 (semi-analytical)) as shown in Fig. (2).

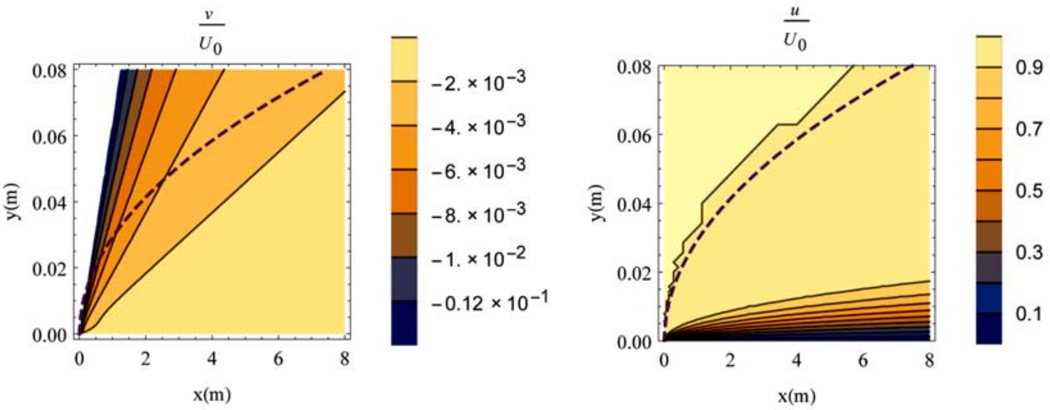

Fig. 3. Visualization of the dimensionless vertical and mainstream velocities near the flat plate (lying in the *xz* plane) in the Blasius regime with $U_0 = 0.1$ *m/s* and $\upsilon = 10^{-6}$ m^2/s.

Eqs. (11) and (1) enable us to provide a closed-form representation of the velocity vector as

$$u(x,y)=U_0\mathrm{erf}\left(\frac{\eta}{\sqrt{2}\,\sigma}\right)\ ;\ v(x,y)=\frac{0.5U_0 y}{x}\left[\mathrm{erf}(\frac{\eta}{\sqrt{2}\,\sigma})-\frac{2}{\sigma^2}\right] \tag{12}$$

Indeed, these formulas are of practical use in terms of performing separate mathematical/programming operators, as opposed to the typical coupled relationship existing in the literature [e.g., Hirsh, 2007, p. 619]

$$(u,v)=\left[U_0 f'\,,\,\frac{1}{2}\sqrt{U_0\upsilon/x}\,(\eta f'-f)\right] \tag{13}$$

An immediate fruit of this approximation is the velocity field visualization, as seen in Fig. (3), with the advantage of eliminating extra computational expense. The plots correspond to $U_0 = 0.1$ m/s and $\upsilon = 10^{-6}$ m^2/s. From the graphs we observe that the crosswise velocity plays a marginal role inside the boundary layer (bellow the dashed-curve) compared with the streamwise component. In fact, computing the boundary layer edge is another outcome of this approximation that will be discussed in subsection (4.1).

4. SOLUTION OUTCOMES

4.1. Determination of Boundary Layer Locus

Our goal here is to identify the boundary layer edge by the Eq. (12). Although there is no orthodox measurement of the viscous region edge due to its asymptotic attenuation (Schlichting, 1979), it is generally accepted to be the locus of points at which the streamwise velocity attains 99% of the potential velocity (Schlichting, 1979, p. 140).

Here, we suggest a new measure $u = 0.999980U_0$, with the reason behind whose appeal becoming apparent in later paragraphs, and base the calculations on the criterion $|\vec{V}|=U_0$, where $|\vec{V}|$ is the magnitude of velocity field. The aim here, therefore, is to solve

$$U_0^2\mathrm{erf}\left(y\sqrt{\frac{U_0}{2\upsilon x\sigma^2}}\right)^2+0.25U_0^2\frac{y^2}{x^2}\left[\mathrm{erf}(y\sqrt{\frac{U_0}{2\upsilon x\sigma^2}})-\frac{2}{\sigma^2}\right]^2=U_0^2 \tag{14}$$

where both sides of equation $|\vec{V}|=U_0$ has been squared. Solving the quadratic Eq. (14) leads to the relation

$$\mathrm{erf}\left(y\sqrt{\frac{U_0}{2\upsilon x\sigma^2}}\right)=\frac{0.5\sqrt{\sigma^4x^2\left[4\sigma^4x^2+\left(\sigma^4-4\right)y^2\right]}+0.5\sigma^2y^2}{\sigma^4\left(x^2+0.25y^2\right)} \tag{15}$$

Since the boundary-layer thickness increases as the square root of the streamwise coordinate (Bird *et al*., 2001, p. 137), we set $y=a\sqrt{x}$ where 'a' is a constant to be determined. As a result, the right-hand side of Eq. (15) is simplified to $\mathrm{erf}\left[a\sqrt{U_0/(2\upsilon\sigma^2)}\right]$, which is constant for any values of parameters involved. For the reason of convenience, we denote $c_0=\mathrm{erf}\left[a\sqrt{U_0/(2\upsilon\sigma^2)}\right]$. Equation (15) then can be written as

$$\left(0.0625a^4c_0^2\sigma^8-0.25a^4c_0\sigma^6+0.25a^4\sigma^4\right)+x\left(0.5a^2c_0^2\sigma^8-a^2c_0\sigma^6-0.25a^2\sigma^8+a^2\sigma^4\right)+x^2\left(c_0^2\sigma^8-\sigma^8\right)=0 \tag{16}$$

Equation (16) holds when the coefficients of variables x^n ($n \in \{0, 1, 2\}$) vanish. The simultaneous solution of this system indicates that $c_0 = 1$ and $\sigma = 1.41$ (note that among the three, only two equations are independent). The constant α

then can be calculated from the definition of c_0. At the same time, it is noted that these values introduce a slight deviation from the expected value $\sigma = 2.18$ corresponding to $f_{2.18}(\eta)$. This issue can be addressed by providing a *smooth* solution to the system (16). By setting σ = 2.18, Eq. (16) is simplified into

$$\left(31.881a^4c_0^2 - 26.833a^4c_0 + 5.646a^4\right)$$
$$+x$$
$$\left(255.048a^2c_0^2 - 107.334a^2c_0 - 127.524a^2 + 22.585a^2\right)$$
$$+x^2\left(510.096c_0^2 - 510.096\right) = 0 \tag{17}$$

It is worth to note that erf(x) is a monotone increasing function with a rapid rate of convergence to unity for large enough arguments ($\lim_{x\to\infty}$ erf(x) =1). A computer algebra package shows that $|\text{erf}(3) - 1| < 5\times10^{-5}$. If an acceptable accuracy is satisfied by 0.005% error at this stage (we shall show that this amount of error results in the negligible 2% error in later equations), we are able to set c_0 = 0.999980. One should bear in mind that c_0 is the dimensionless velocity u/U_0 combined with the assumption that boundary layer locus is a parabola. This demonstrates that the intuitive criterion $u = 0.99U_0$ in the boundary layer literature (Schlichting, 1979; Bird *et al.*, 2011) has been changed to $u = 0.999980U_0$ in this work.

With this attribution, the coefficients of x and x^0 become of orders $O(a^2)$ and $O(a^4)$ respectively, being negligible on the level of reality in which 0.1 $m/s < U_0 < 100$ m/s and $\upsilon = 10^{-6}$ m²/s. Therefore, the predominant source of error is the coefficient of x^2, with a nearly 2% error (acceptably small). This verifies that the assumption of parabolic growth of y versus x still remains valid for $\sigma = 2.18$.

The final outcome, therefore, is the determination of boundary layer locus as

$$y = 9.24895\sqrt{\frac{\upsilon x}{U_0}} \tag{18}$$

where a has been derived by $\sqrt{2\upsilon\sigma^2/U_0}\,\text{erf}^{-1}(c_0)$. It should be mentioned that this result is in agreement with the results in Schlichting (1979) in terms of dimension and relation among the parameters involved. However, the value suggested in Eq. (18) is nearly twice higher than Schlichting (1979, p. 140) ($\delta(x) = 5\sqrt{\upsilon x/U_0}$). The distinction mainly arisen from the fact that in Schlichting (1979) $u = 0.99U_0$ is accepted as a premise without any recourse to mathematical analysis. Although setting $u = 0.99U_0$ here results in an approximately the same factor as that in Schlichting (calculated to be $y = 5.6\sqrt{\upsilon x/U_0}$), the current outcomes, being on the grounds of merely three assunptions $|\vec{V}| = U_0$, $y \propto \sqrt{x}$ and $u = 0.999980U_0$, are believed to provide more reasonable estimations of the viscous zone. In Fig. 3, we have plotted the boundary layer parabola by the dashed-line.

4.2. *Momentum and Energy Diffusion*

Our goal here is to infer the momentum and energy conservation equations in the diffusion modes from the previous approximation. A direct calculation using Eq. (11) in conjunction with the chain rule shows that

$$\frac{\partial u}{\partial x} = D_b\frac{\partial^2 u}{\partial y^2} \tag{19}$$

where $D_b = \sigma^2\upsilon/(2U_0)$ represents the effect of mollecular mometum flux in the laminar visous zone. Since this equation is valid for the absence of pressure gradient, we call it the *uniform-pressure momentum constant*. Equation (19) describes a stationary diffusion process in the 2-dimensional spatial domain (the reader can compare with the typical time-dependent heat diffusion in Logan (2004)). Here, the constant of diffusion (D_b) depends on the viscosity and the upstream velocity; the more the fluid is viscous and its flow retarded, the more the momentum flux transpires in the control volume. This leads us to a significant inference that the transport of momentum in the laminar boundary layer is governed by a *diffusion* process.

Furthermore, applying the chain rule to Eq. (6) leads to

$$\frac{\partial T}{\partial x} = -0.5\frac{y}{x}\frac{\partial T}{\partial y} \tag{20}$$

Solving this equation for y/x and substituting to the vertical and mainstream velocity components in Eq. (12) shows that

$$v(x,y) = -U_0\frac{\partial T/\partial x}{\partial T/\partial y}\left[\frac{u(x,y)}{U_0} - \frac{2}{\sigma^2}\right]. \tag{21}$$

Finally, combination of Eq. (21) and Eq. (4) results in the differential equation

$$\frac{\partial T}{\partial x} = D_t\frac{\partial^2 T}{\partial y^2} \tag{22}$$

where $D_t = \alpha\sigma^2/(2k_0U_0)$ is the *uniform-pressure thermal constant* since it represents how heat diffuses between the plate and the physical infinity in the paucity of pressure gradient. Indeed, the increased radiation parameter and free stream velocity would be impediments to the diffusion of heat, while the thermal diffusivity is proportionally correlated, predicting a linearly grown flux as a consequence of a steady improvement in the thermal diffusivity value.

These results conform well with Hirsch (2007, p. 96) demonstrating that the Navier-Stokes equations, as a consequence of boundary layer approximations, will reduce to relationships very close to standard parabolic second order PDEs. However, to the authors' knowledge, this study is the first one that explicitly presents the momentum/energy laws in the Blasius regime in the form of diffusion equations. Interestingly, by dividing the *uniform-pressure momentum constant* by the *uniform-pressure thermal constant*, we attain an indication for the boundary layer relative readiness to transport momentum/energy. This ratio introduces a modified Prandtl number

$$\breve{\Pr} = \frac{D_b}{D_t} = k_0 \frac{\upsilon}{\alpha} \quad (23)$$

which is analogous with the definition of Prandtl number $\Pr = \frac{\upsilon}{\alpha}$ for a typical flow system (Bird *et al*., 2001, p. 268). However, the modified form of Prandtl number includes the intrinsic effect of radiation parameter as well. It is noted that for large values of radiation parameter, the $\breve{\Pr}$ asymptotes to $\Pr$.

Finally, we support the validation of some of our results by means of comparison against higher level approximations seen in Bataller (2008). If we substitute Eq. (11) for $f(\eta)$ in Eq. (5), there appears a linear second-order ODE having θ as the only unknown involved. This can represent a remarkable simplification in the solving process since the calculation of θ becomes decoupled from the calculation of $f(\eta)$. Table (1) shows the solution of this ODE, representing the value of $\theta'(0)$ in the current work and Bataller (2008), which the former has been obtained by typical numerical methods in ordinary differential equations. A good agreement is observed between the results. This also suggests that the problem of thermal radiation would be more tractable, in terms of investigating its physics, if we benefit from this or similar semi-analytical approximations.

Table 1 Comparison of the solutions to Eq. (5) obtained in this work and in Bataller (2008) with $K_0 = 1$

$\breve{\Pr}$	$\theta'(0)$	
	Bataller (2008)	Current Study
0.1	-	0.14035
0.4	-	0.24094
0.7	0.29268	0.29607
5	0.57669	0.59023
10	0.72814	0.74735
50	1.24729	1.28518
100	1.57183	1.62100

5. CONCLUSIONS

In this study, we investigated the basic properties of laminar flow regime near a flat plate governed by the Blasius ODE. By providing a semi-analytical approximation, we calculated the thickness of viscous region. It was shown that, due to the amended criteria suggested in this work, the boundary layer edge expands to include almost 85% more volume in the vicinity of the flat plate. We also introduced a *uniform-pressure thermal constant* and a *uniform-pressure momentum constant* for the two suggested energy and momentum time-independent equations.

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