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Dispersion in Chiral Fluid in the Presence of Convective Current between Two Parallel Plates Bounded by Rigid Permeable Walls

N. Rudraiah[1, 2†] and S.V. Raghunatha Reddy[1]

[1]*UGC-Centre for Advanced Studies in Fluid Mechanics, Dept. of Math, Bangalore University. Bangalore-560 001, India*
[2]*National Research Institute for Applied Math., #462/G, 7th Cross, 7th Block(West), Jayanagar, Bangalore-560 070,India*

†*Corresponding Author Email: rudraiahn@hotmail.com*

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ABSTRACT

This paper describes the use of Taylor dispersion analysis to study the dispersion of chiral fluid flow in a channel in the presence of the convective current bounded by rigid permeable walls. Analytical solution for velocity in the presence of a transverse magnetic field is obtained and it is computed for different values of electromagnetic parameter Wem. The results reveal that the velocity increases with an increase in the electromagnetic parameter, Wem. Concentration distribution is also determined analytically in the presence of advection of concentration of chiral fluid. It is shown that the molecules of chiral fluid dispersed relative to the plane moving with the mean speed of flow with an effective dispersion coefficient, D^*, called Taylor dispersion coefficient. This is numerically computed for different values of electromagnetic parameter, Wem, Paclet number Pe and Reynolds number, Re. The results shows that dispersion coefficient, D^* decreases monotonically with Reynolds number, Re, Paclet number Pe, but increase with an increase in electromagnetic parameter, Wem.

Keywords: Chiral Fluid, Lorentz force with chirality parameter, Convection current, Coronary artery diseases, Synovial joints, Dispersion.

NOMENCLATURE

u	Velocity component in the x-direction	$\vec{E}$	Electric field
v_0	Suction velocity in the y-direction	ε	Dielectric constant
h	Height of the channel	γ	Chirality coefficient
B_0	Applied magnetic field in the z-direction	p	Pressure
μ	Magnetic permeability	Re	Reynolds number
μ_f	Coefficient of viscosity of the fluid	Pe	Peclet number
J	Current density	Wem	Electromagnetic parameter
ρ	Density of the fluid	C	Concentration of species
$\vec{D}$	Dielectric field	D^*	Dispersion coefficient
ρ_e	The density of charge distribution		

1. INTRODUCTION

In recent years, considerable interest has been evinced in the development of new technologies like Information Technology, Bio-Technology, Nano-Technology, technologies involving Smart and Chiral Materials, using improved and novel processing routes which will replace most of the existing technologies today.

These are going to change every aspect of our lives and lead to generation of new capabilities, new materials and new products. An important aspect, associated with these new technologies, is their multi disciplinary nature with applications in Science, Engineering and Technology and their impact on society is expected to be wide spread and all pervasive. By definition, a three dimensional object is chiral if it cannot be brought into congruence with its mirror image by any amount of translation and rotation.

In other words chiral Fluid is a fluid in which the molecules have the property of handedness and must be either right handed or left handed. Therefore, chirality is connected with handedness (Davankov, 1997, 1983). The fluids like sugar solution, turpentine, glucose, drugs, carbohydrates, proteins, nutrients, amino acids, RBC, WBC, enzymes in our body cells, ntibodies, hormones, body-fluids and so on (see Sharpless *et al,* 1997, 2001, Anet, 1983 and Nasipuri, 2004, Chen *et al*, 2004), exhibit chirality. Proper functioning of artificial organs like Synovial Joints (SJs) and Coronary Artery Diseases (CAD) in biomedical engineering depend on the dispersion of hyaluronic acid and nutrients in synovial fluid and the dispersion of RBC, WBC and so on in physiological fluids in arteries. Dispersion phenomenon is useful in Purification of sugar solution in sugar industry because sugar solution is a chiral fluid. It is also useful in the design of an efficient antenna and so on.

Literature is available on theoretical and experimental aspects of solid chiral materials (see Arago, 1811, Biot, 1812, Jaggard, 1979, Varadan and Varadan, 1989, and Laktakia, 1985, 1994). However, much attention has not been given to a detailed study of dispersion in chiral fluids in spite of its importance in many practical problems cited above. The study of it is the main objective of the present paper.

To achieve the objective of this paper, the required basic equations for two dimensional flow together with Maxwell's equations and required constitutive equations for chiral fluids are given in section 2. Using these basic equations with suitable approximations, the required velocity and the Taylor dispersion coefficient are determined in the presence of transverse magnetic field and distribution of charge density is decreasing continuously with height are obtained in section 3. The results, discussion and conclusion are given in the final section 4.

2. MATHEMATICAL FORMULATION

We consider the physical configuration as shown in Fig.1 which consists of a chiral fluid flow through a rectangular channel bounded by rigid and permeable walls at y = _h and x-axis is parallel to the plates, y and z axes are perpendicular to it. We deal with two-dimensional chiral fluid flow with u and v the components velocity in the x and y directions respectively and a uniform applied magnetic field B0 in the z direction. We assume the chiral fluid to be incompressible, viscous and Newtonian and the flow is governed by modified Navier-Stokes equation (modification means the inclusion of Lorentz force with chirality parameter, (see Anet, 1983 and Nasipuri, 2004, Sharpless, 2001, Varadan, 1989). Therefore the governing equations describing a chiral fluid flow in a channel are:

The conservation of mass

$$\nabla \cdot \vec{q} = 0 \tag{1}$$

The conservation of momentum

$$\rho\left(\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q}\right) = -\nabla p + \mu_f \nabla^2 \vec{q} + \vec{J}\times\vec{B} \tag{2}$$

the conservation of species

$$\frac{\partial C}{\partial t} + (\vec{q}\cdot\nabla)C = D\left(\nabla^2 C\right) \tag{3}$$

the conservation of electric charges

$$\frac{\partial \rho_e}{\partial t} + \nabla\cdot\vec{J} = 0 \tag{4}$$

These equations have to be supplemented with the Maxwell's equations

$$\nabla\cdot\vec{D} = \rho_e \tag{5}$$

$$\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{6}$$

$$\nabla\times\vec{H} = \vec{J} \tag{7}$$

$$\nabla\cdot\vec{B} = 0 \tag{8}$$

together with the constitutive equations for chiral fluids (Varadan *et al.*, 1989, and Rudraiah *et al.*, 2000)

$$\vec{D} = \varepsilon\vec{E} + i\gamma\vec{B} \tag{9}$$

$$\vec{B} = \mu\vec{H} - i\mu\gamma\vec{E} \tag{10}$$

$$\vec{J} = \rho_e\vec{q} + \frac{\partial\vec{D}}{\partial t} \tag{11}$$

Here $\vec{q} = (u, v)$ is the velocity, $\vec{B}$ the magnetic induction, $\vec{H}$ the magnetic field, $\vec{J}$ the current density, $\vec{D}$ the dielectric field, $\vec{E}$ the electric field. p the pressure, ρ the density of the fluid, ρ_e the distribution of electric charge density, μ the magnetic permeability, ε the dielectric constant , γ the chirality coefficient, $\rho_e\vec{q}$ the convective current, $\partial\vec{D}/\partial t$ the displace current in the absence of conduction current and $\vec{J}\times\vec{B}$ is the Lorentz force. In this paper we consider only the convective current. For the chosen physical configuration as shown in Fig 1, and using the above assumptions, the required basic equations, in Cartesian form for chiral fluid, are

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \rho_e B_0 v \tag{12}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}$$
$$+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-\rho_e B_0 u \tag{13}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{14}$$

$$\frac{\partial \rho_e}{\partial t}+u\frac{\partial \rho_e}{\partial x}+v\frac{\partial \rho_e}{\partial y}=0 \tag{15}$$

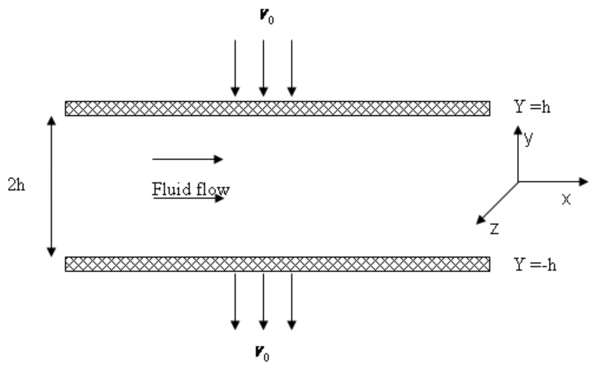

Fig. 1. Physical configuration

Following Taylor (1953), we consider the flow to be steady, unidirectional, fully developed flow and parallel to the plates in the x direction, such that

$$u=u(y),\ \frac{\partial}{\partial t}=0,\ v=v_0,\ \rho_e=\rho_e(y) \text{ and}$$

$$\frac{\partial p}{\partial x}=constant \tag{16}$$

under these approximations the above Eqs. (12) and (13) become

$$\rho v_0\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2}+\rho_e B_0 v_0 \tag{17}$$

$$0=-\frac{\partial p}{\partial y}-\rho_e B_0 u \tag{18}$$

Equation (17) revels that an interaction of magnetic field with the fluid is due to the suction velocity. The Eqs. (17) and (18) are made dimensionless using

$$x^*=\frac{x}{h},\ \eta=\frac{y}{h},\ p^*=\frac{p}{\rho v_0^2},\ u^*=\frac{u}{v_0} \text{ and}$$

$$\rho_e^*=\frac{\rho_e h^2}{\varepsilon V} \tag{19}$$

where the asterisk (*) denote the dimensionless quantities, h the characteristic height, V the Electric potential and other quantities are as defined in the Eqs.(10 to 11). Therefore the dimensionless form of the above Eqs. (17) and (18), using eq. (19) and for simplicity neglecting the asterisks, take the form

$$\mathrm{Re}\frac{\partial u}{\partial \eta}=k_2+\frac{\partial^2 u}{\partial \eta^2}+k_1\rho_e \tag{20}$$

$$0=\frac{\partial P}{\partial \eta}+\frac{k_1}{R_e}u\rho_e \tag{21}$$

Where $k_1=Wem\,\mathrm{Re}$, $k_2=-\mathrm{Re}\,P$, $P=\frac{\partial p}{\partial x}$ $Wem=\frac{\varepsilon V B_0}{h\rho v_0}$ the electromagnetic parameter and $\mathrm{Re}=\frac{h v_0}{\nu}$ the suction Reynolds number . We assume (Rudraiah *et al*, 2011) that the density of charge distribution, ρ_e, in chiral fluid decreases continuously in the vertical direction of the form

$$\rho_e=\rho_{eo}e^{-\beta\eta} \tag{22}$$

Where β is the charge density stratification factor. Therefore, eq. (20), using eq. (22), takes the form

$$\mathrm{Re}\frac{\partial u}{\partial \eta}=k_2+\frac{\partial^2 u}{\partial \eta^2}+k_1 e^{-\beta\eta} \tag{23}$$

Satisfying the no slip boundary conditions

$$u=0 \text{ at } \eta=\pm 1 \tag{24}$$

Equation (23) is solved analytically using the boundary conditions eq. (24) and obtained

$$u=k_5\eta-k_6 e^{-\beta\eta}-k_3 e^{\mathrm{Re}\,\eta}+k_4 \tag{25}$$

where the constants k_3 to k_6 are given in the appendix.

3. DISPERSION MODEL

If C is the concentration of a chiral fluid such as proteins, nutrients, amino acids, carbohydrates and sugar (see Anet, 1983 and Nasipuri, 2004) in physiological fluid, regarded as chiral fluid and diffuses in a fully developed flow given by Eq. (3), then C satisfies the advection-diffusion equation. In the Taylor dispersion mechanism, one has to consider quasi steady flow involving the chiral particles to understand the hydrodynamic dispersion. Following Taylor (1953), we assume that the longitudinal diffusion is much smaller than the transverse diffusion and the diffusivity D_m, viscosity μ_f, and the pressure gradient P are assumed to be constants. Under these approximations the advection of concentration of chiral fluid satisfies the equation

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v_0\frac{\partial C}{\partial y}=D_m\frac{\partial^2 C}{\partial y^2} \tag{26}$$

where D_m is the molecular diffusivity. This equation is made dimensionless using the non-dimensional quantities

$C^* = \dfrac{C}{C_0},\ \tau_1 = \dfrac{t}{\bar{t}},\ \bar{t} = \dfrac{L}{\bar{u}},\ x^* = \dfrac{x}{L},\ y^* = \dfrac{y}{h}$ and the moving coordinate $\xi = \dfrac{x - \bar{u}t}{L}$ (27)

where L is the characteristic length and $\bar{u}$ is the average velocity of Eq. (25). The Taylor dispersion deals with advection across the plane moving with the mean speed $\bar{u}$ and is given by

$$\bar{u} = \frac{1}{2}\int_{-1}^{1} u\, d\eta = -\frac{k_6}{\beta} \sinh\beta + \frac{k_3}{\mathrm{Re}} \sinh \mathrm{Re} + k_4 \qquad (28)$$

Substituting Eqs. (27) and (28) into Eq. (26), we get

$$\frac{1}{\bar{t}}\frac{\partial C}{\partial \tau_1} + \frac{W(\eta)}{L}\frac{\partial C}{\partial \xi} + Pe\frac{\partial C}{\partial \eta} = \frac{D}{h^2}\frac{\partial^2 C}{\partial y^2} \qquad (29)$$

where $W(\eta) = u - \bar{u}$

$$W(\eta) = -k_5\eta - k_6 e^{-\beta\eta} - k_3 e^{\mathrm{Re}\,\eta} + k_8 z \qquad (30)$$

Following Taylor (1953), if the Taylor longitudinal condition is valid even in the present problem, then the partial equilibrium at any cross-section of the layer is assumed to be valid and the variation of C with y is obtained from Eq. (29) and it is of the form

$$\frac{\partial^2 C}{\partial \eta^2} - Pe\frac{\partial C}{\partial \eta} = \frac{W(\eta) h^2}{DL}\frac{\partial C}{\partial \xi} \qquad (31)$$

We solve this equation using permeable wall conditions

$$C = 1 \text{ at } \eta = \pm 1 \qquad (32)$$

The solution of the Eq. (31), satisfying the conditions Eq. (32) (Rudraiah *et al.*, 1977) is

$$C = 1 + \psi\left[\begin{array}{l} k_{11}\eta^2 + k_{12}\eta + k_{13}e^{Pe\eta} + k_9 e^{-\beta\eta} \\ + k_{10}e^{\mathrm{Re}\,\eta} + k_{14} \end{array}\right] \qquad (33)$$

Where $\psi = \dfrac{h^2}{DL}\dfrac{\partial C}{\partial \xi}$ (34)

and the constants k_i ($i = 9 to 14$) are mentioned in the appendix. The volumetric rate M at which the solute is transported across a section of the layer of unit breadth is

$$M = \int_{-1}^{1} C\, W(\eta)\, d\eta \qquad (35)$$

This, using Eqs. (30) and (33), performing the integration and after simplification, becomes

$$M = -F\psi \qquad (36)$$

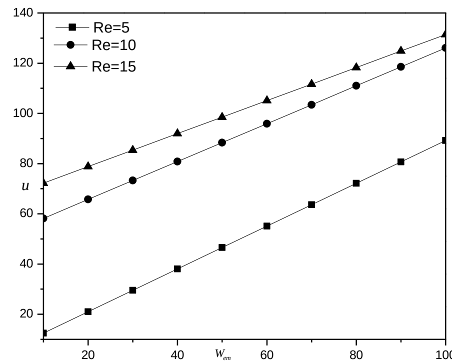

Fig. 2. Variation of u v/s W_{em} for different values of R_e=5, 10 and 15

Where

$$\begin{aligned} F = [&k_7k_{12}\left(\frac{2\cosh\beta}{\beta^2} - \frac{2\sinh\beta}{\beta^3} - \frac{\sinh\beta}{\beta}\right) \\ &+ k_4k_{12}\left(\frac{2\cosh \mathrm{Re}}{\mathrm{Re}^2} - \frac{2\sinh \mathrm{Re}}{\mathrm{Re}^3} - \frac{\sinh \mathrm{Re}}{\mathrm{Re}}\right) \\ &+ 2\left(k_7k_{13} - k_6k_{10}\right)\left(\frac{\cosh\beta}{\beta} - \frac{\sinh\beta}{\beta^2}\right) \\ &+ 2\left(k_4k_{13} - k_6k_{11}\right)\left(\frac{\sinh \mathrm{Re}}{\mathrm{Re}^2} - \frac{\cosh \mathrm{Re}}{\mathrm{Re}}\right) \\ &+ 2k_6k_{14}\left(\frac{\cosh Pe}{Pe} - \frac{\sinh Pe}{Pe^2}\right) - \frac{2}{3}\left(\frac{k_8k_{12}}{2} + k_6k_{13}\right) \\ &- 2k_7k_{14}\frac{\sinh(Pe - \beta)}{Pe - \beta} - 2k_4k_{14}\frac{\sinh(\mathrm{Re} + Pe)}{\mathrm{Re} + P} \\ &+ 2k_8k_{14}\frac{\sinh Pe}{Pe} + k_7k_{10}\frac{\sinh 2\beta}{\beta} \\ &- 2(k_4k_{10} + k_7k_{11})\left(\frac{\sinh(Pe - \beta)}{Pe - \beta}\right) \\ &- 2(k_7k_{15} - k_8k_{10})\frac{\sinh\beta}{\beta} - k_4k_{11}\frac{\sinh 2\mathrm{Re}}{\mathrm{Re}} \\ &- 2(k_4k_{15} - k_8k_{11})\frac{\sinh \mathrm{Re}}{\mathrm{Re}} + 2k_8k_{15}] \end{aligned} \qquad (37)$$

Further, following Taylor (1953), we assume that the variations of C with y is small compared with those in the longitudinal direction and if C_m the mean concentration over a section, then $\dfrac{\partial c}{\partial t}$ is indistinguishable from $\dfrac{\partial c_m}{\partial t}$ so that eq. (36), using eq. (34) may be written as

$$M = -F\frac{h^3}{DL}\frac{\partial C_m}{\partial \xi} \qquad (38)$$

This shows that C_m is displaced relative to a plane which moves with the mean velocity which is exactly as if it would have been diffused by a process which obeys the same law as molecular diffusion but with a relative diffusion coefficient D^*, called Taylor dispersion coefficient. The fact that no material is lost in the process is expressed, following Taylor (1953), by the continuity equation for C_m given by

$$\frac{\partial M}{\partial \xi} = -\frac{1}{L}\frac{\partial C_m}{\partial t} \tag{39}$$

where the time is made dimensionless using the scale $\frac{L}{\bar{u}}$, $\bar{u}$ is characteristic velocity. Using Eqs. (38) and (39) becomes

$$\frac{\partial C_m}{\partial t} = D^* \frac{\partial^2 C_m}{\partial \xi^2} \tag{40}$$

Where $D^* = \frac{h^2 \bar{u}}{D} F$ (41)

Where F is given by eq. (37, $Pe = \frac{v_0 h}{D}$ is the Peclet number. The D^* given by eq. (41) is computed for different values of the suction Reynolds number Re ,

Peclet number Pe , the electro magnetic parameter Wem and the results are depicted graphically in Figures 3 and 4.

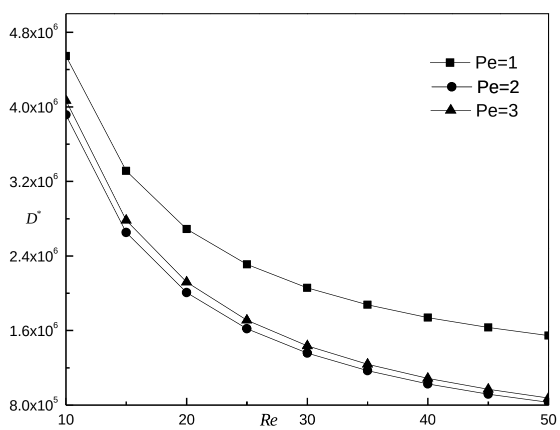

Fig. 3. Variation of D^* v/s R_e for different values of P_e=1, 2 and 3

4. RESULTS AND DISCUSSION

The variation of velocity versus electro magnetic parameter Wem for different values Reynolds number is shown in figure 2. From this figure it is evident that the velocity increases with an increase in electro magnetic parameter. Physically this is attributed due to the fact that electric field introduces small scale turbulence.

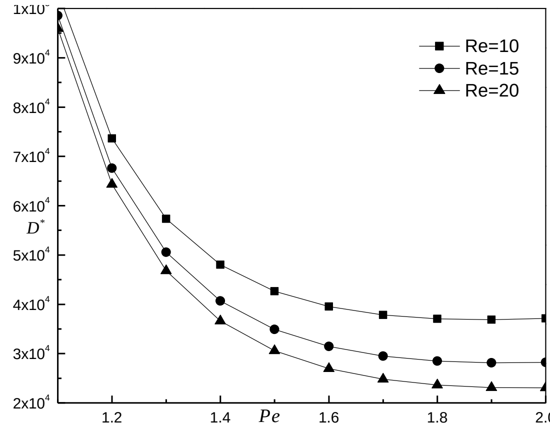

Fig. 4. Variation of D^* v/s P_e for different values of R_e=10, 15 and 20

The variation of dispersion coefficient D^* versus Reynolds number for different values Peclet number and for a fixed value of Wem is shown in the figure 3. Form this figure it is evident that the dispersion coefficient decreases monotonically with increasing Reynolds number, Re and Peclet number, Pe . This is because of increase in the suction velocity. The variation of dispersion coefficient D^* versus Peclet number for different values Reynolds number and for a fixed value of Wem is shown in the figure 4. Form this figure it is evident that the dispersion coefficient decreases monotonically with an increasing Peclet number. Form figure 5 it revels that mass flow rate versus suction Reynolds number for different values of paclet number, it shows that as paclet number increases the mass flow rate decreses it is due suffresion of velosity because of suction reynolds number.

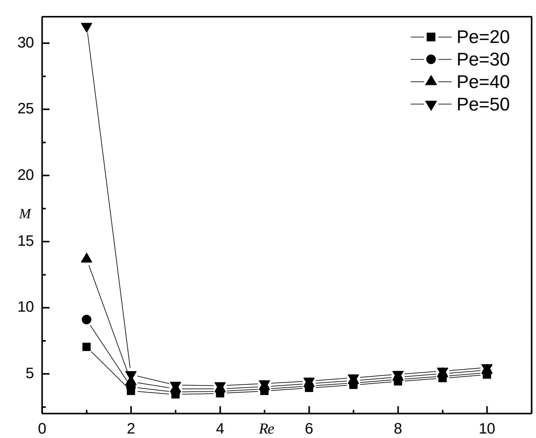

Fig. 5. Variation of Concentration M versus Reynolds number for different values of Pe =20, 30, 40 and 50

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APPENDIX

$$k_3 = co\sec h\,\mathrm{Re}\left(k_5 + k_1 k_7 sinh\,\beta\right)$$

$$k_4 = -k_1 k_7 \cosh\beta - \coth\mathrm{Re}\left(k_5 + k_1 k_7 sinh\,\beta\right)$$

$$k_5 = \frac{k_1}{\mathrm{Re}}$$

$$k_6 = \frac{k_1}{k_7}$$

$$k_7 = \frac{1}{\beta(\beta + \mathrm{Re})}$$

$$k_8 = \frac{k_6}{\beta}\sinh\beta + \frac{k_3}{\mathrm{Re}}\sinh\mathrm{Re}$$

$$k_9 = -k_6 k_{16}$$

$$k_{10} = -k_3 k_{15}$$

$$k_{11} = -\frac{k_5}{2Pe}$$

$$k_{12} = -\left(\frac{k_5}{Pe^2} + \frac{k_8}{Pe}\right)$$

$$k_{13} = \frac{k_5 Co\sec hPe}{Pe^2} + k_9 Sinh\beta.CosechPe + k_{10} Co\sec hPe.Sinh\,\mathrm{Re} - \frac{k_8 CosechPe}{Pe}$$

$$k_{14} = \frac{k_5}{2Pe} - \frac{CothPe}{Pe^2}k_5 - k_9 Co\sec hPe.Sinh(\beta + Pe) - k_{10} Co\sec hRe.Sinh(Pe - \mathrm{Re}) - k_8\frac{CothPe}{Pe}$$

$$k_{15} = \frac{1}{Re(Re - Pe)}$$

$$k_{16} = \frac{1}{\beta(\beta + \mathrm{Re})}$$