Journal of Applied Fluid Mechanics, Vol. 8, No. 2, pp. 223-229, 2015.
Available online at www.jafmonline.net, ISSN 1735-3572, EISSN 1735-3645.
DOI: 10.18869/acadpub.jafm.67.221.21329

Flow and Heat Transfer of an Exponential Stretching Sheet in a Viscoelastic Liquid with Navier Slip Boundary Condition

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(Received July 14, 2013; accepted March 16, 2014)

ABSTRACT

Viscoelastic boundary layer flow and heat transfer over an exponential stretching continuous sheet have been investigated in this paper. Numerical solution of the highly non-linear momentum equation and heat transfer equation are obtained. Two cases are studied in heat transfer, namely (i) the sheet with prescribed exponential order surface temperature (PEST case) and (ii) the sheet with prescribed exponential order heat flux (PEHF case). The governing coupled, non-linear, partial differential equations are converted into coupled, non-linear, ordinary differential equations by a similarity transformation and are solved numerically using shooting method. The classical explicit Runge-Kutta-Fehlberg 45 method is used to solve the initial value problem by the shooting technique. The effects of various parameters such as viscoelastic parameter, slip parameter, Eckert number and Prandtl number on velocity and temperature profiles are presented and discussed. The results have possible technological applications in the liquid-based systems involving stretchable materials.

Keywords: Stretching sheet, Slip parameter, Prandtl number, Eckert number, Shooting Method.

NOMENCLATURE

- A_0, A_1 prescribed constants
- C_p specific heat at constant pressure
- E eckert number
- k thermal conductivity
- k_0 viscoelastic parameter
- k_1^* dimensionless viscoelastic parameter
- L reference length
- Pr prandtl number
- Re reynolds number
- T fluid temperature of the moving sheet
- T_w wall temperature
- T_∞ temperature far away from the sheet
- U_0 constant
- U_w stretching velocity of the boundary
- u, v velocity components along x and y direction
- x flow directional co-ordinate along the stretching sheet
- y distance normal to the stretching sheet
- X, Y dimensionless co-ordinates

Greek symbols

- α_1 slip parameter
- α^* non-dimensional slip parameter
- ν kinematic viscosity
- μ dynamic viscosity
- θ dimensionless temperature in PEST case
- ϕ dimensionless temperature in PEHF case
- ρ density of the fluid
- ψ dimensionless stream function

Subscripts

- w wall temperature
- ∞ ambient temperature condition

1. INTRODUCTION

Boundary layer flow on continuous moving surface is an important type of flow occurring in a number of engineering processes. Aerodynamic extrusion of plastic sheets, cooling of an infinite metallic plate in a cooling path, the boundary layer along a liquid film in condensation process and a polymer sheet of filament extruded continuously from a die are examples of practical applications of continuous moving surfaces. Gas blowing, continuous casting and spinning of fibers also involve the flow due to a stretching surface.

"Sakiadis (1961 a, b, c)" initiated the study of the boundary layer flow over a continuous solid surface moving with constant speed. "Erickson *et al*. (1969)" extended the work of Sakiadis to account for mass transfer at the stretching sheet surface. "Tsou *et al*. (1967)" reported both analytical and experimental results for the flow and heat transfer aspects developed by a continuously moving surface. "Crane (1970)" studied the steady two dimensional boundary layer flow caused by the stretching sheet, which moves in its own plane with a velocity which varies linearly with the axial distance. Several researchers considered various aspects of momentum and heat transfer characteristics in boundary layer flow over a stretching boundary ("Stokes (1966)", "Gupta and Gupta (1977)", "Rajagopal *et al*. (1984)", Siddappa and Abel (1985)", "Andersson (1992)", "Kumaran and Ramanaiah (1996)", "Cortell (2007)", Sekhar and "Chethan (2010, 2011)").

"Magyari and Keller (2000)" studied the heat and mass transfer on the boundary layer flow due to an exponentially stretching surface. "Elbashbeshy (2001)" added new dimension to the study on exponentially stretching surface. "Partha *et al*. (2004)" have examined the mixed convection flow and heat transfer from an exponentially stretching vertical surface in quiescent liquid using a similarity solution. Heat and mass transfer in a viscoelastic boundary layer flow over an exponentially stretching sheet were investigated by "Khan and Sanjayanand (2005, 2006)". "Sajid and Hayat (2008)" considered the influence of thermal radiation on the boundary layer flow due to an exponentially stretching sheet. "Bidin and Nazar (2009)" studied the steady two dimensional boundary layer flow and heat transfer of an incompressible viscous fluid in the presence of thermal radiation over an exponentially stretching sheet. "Sekhar and Chethan (2012)" analyzed the flow and heat transfer due to an exponentially stretching continuous surface in the presence of Boussinesq-Stokes suspension. "Siddheshwar *et al*. (2014)" extended this work to study the flow and heat transfer characteristics in the presence of a transverse uniform magnetic field.

A common feature of all these analyses is the assumption that the flow field obeys the conventional no-slip condition at the sheet. But there are situations wherein such condition is not appropriate. Especially, no-slip condition is inadequate for most non-Newtonian fluids. For example, polymer melts often exhibit macroscopic wall slip and that in general is governed by a non-linear and monotone relation between the slip velocity and traction. The fluids exhibiting boundary slip find applications in technology such as in the polishing of artificial heart valves and internal cavities. "Navier (1827)" suggested a slip boundary condition in terms of linear shear stress. Therefore the present work has been undertaken in order to analyze the flow and heat transfer characteristics due to an exponentially stretching sheet in the presence of a viscoelastic fluid with slip effects.

2. MATHEMATICAL FORMULATION

We consider a steady, two-dimensional boundary layer flow of an incompressible second order viscoelastic fluid over a stretching sheet for analysis. Boundary is assumed to be moving axially with a velocity of exponential order in distance by applying two equal and opposite forces along the x-axis by keeping the origin fixed. Since the fluid under consideration is viscoelastic, the energy will be stored in the fluid by means of frictional heating due to viscous dissipation. So we take account of this. However, we assume that the fluid possesses strong viscous property in comparison with the elastic property. With this assumption we neglect the contribution of heat due to elastic deformation. The governing boundary layer equations for momentum and heat transfer in such flow situations are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-k_0\left\{u\frac{\partial^3 u}{\partial x\partial y^2}+v\frac{\partial^3 u}{\partial y^3}-\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right\}, \tag{2.2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho C_p}\frac{\partial^2 T}{\partial y^2}+\frac{\mu}{\rho C_p}\left(\frac{\partial u}{\partial y}\right)^2, \tag{2.3}$$

subject to the boundary conditions:

$$\begin{gathered}u=U_w(x)=U_0 e^{\frac{x}{L}}+\alpha_1\nu\frac{\partial u}{\partial y},\ v=0,\\ \left.\begin{cases}T=T_w=T_\infty+A_0 e^{\frac{2x}{L}} & in\ PEST\ case\\ -k\left(\dfrac{\partial T}{\partial y}\right)_w=A_1 e^{\frac{5x}{2L}} & in\ PEHF\ case\end{cases}\right\}at\ \ y=0,\\ u\to 0,\ T\to T_\infty\ \ as\ \ y\to\infty.\end{gathered} \tag{2.4}$$

where u and v are the velocity components of the fluid in x and y directions, μ is the viscosity, ν is the kinematic coefficient of viscosity, k_0 is the viscoelastic parameter, U_w stands for stretching velocity of the boundary, U_0 is a constant, L is the reference length, α_1 is the slip parameter, ρ is the density, k is the thermal conductivity, T is the temperature, T_w is the temperature at the wall, T_∞ is the temperature outside the dynamic region and C_p is the specific at constant pressure. Here A_0 and A_1 are the parameters of the temperature distribution on the stretching surface.

We introduce the stream function $\psi(x, y)$ defined by

$$u=\frac{\partial\psi}{\partial y},\ v=-\frac{\partial\psi}{\partial x} \tag{2.5}$$

The above set of partial differential equations is converted into a set of ordinary differential equations using the following similarity transformation.

$$X=\frac{x}{L},Y=\frac{y}{L},\Psi(X,Y)=\frac{\psi(x,y)}{\nu}=\sqrt{2Re}\,f(\eta)e^{\frac{X}{2}},\eta=Y\sqrt{\frac{Re}{2}}e^{\frac{X}{2}},$$

$$\left\{\begin{array}{ll}\theta(\eta)=\dfrac{T-T_\infty}{T_w-T_\infty} & in\ PEST\ case\\ \phi(\eta)=\dfrac{T-T_\infty}{\dfrac{A_1}{k}\sqrt{\dfrac{2}{Re}}e^{2X}} & in\ PEHF\ case\end{array}\right\}. \quad (2.6)$$

where η is the similarity variable and $Re=\frac{U_0 L}{\nu}$ is the Reynolds number.

$$f'''+f\,f''-2(f')^2-k_1^*\left\{3f'\,f'''-\frac{1}{2}f\,f''''-\frac{3}{2}f''^2\right\}=0, \quad (2.7)$$

The boundary conditions (2.4) for velocity can be written as:

$$f(0)=0,\ f'(0)=1+\alpha^* f''(0),$$
$$f'(\infty)\to 0,\ \ f''(\infty)\to 0. \quad (2.8)$$

Here $k_1^*=\frac{k_0 U_w}{\nu}$ is the dimensionless viscoelastic parameter and $\alpha^*=\alpha_1\sqrt{\frac{U_w \nu}{2L}}$ is the non-dimensional slip parameter.

Using equation (2.6) in equations (2.3) and (2.4), we get:

(i) PEST:

$$\theta''+Pr\left(f\,\theta'-4f'\theta\right)=-Pr\,E\,f''^2, \quad (2.9)$$
$$\theta(0)=1,\theta(\infty)\to 0. \quad (2.10)$$

where $E=\frac{U_0^2}{T_0 C_p}$ is the Eckert number.

(ii) PEHF:

$$\phi''+Pr\left(f\,\phi'-4f'\phi\right)=-Pr\,E\,f''^2, \quad (2.11)$$

$$\phi'(0)=-1,\phi(\infty)\to 0. \quad (2.12)$$

where $E=\frac{U_0^2}{A_1 C_p\sqrt{\frac{2}{Re}}}$ is the Eckert number.

We now outline the procedure for solving two boundary value problems (2.9)-(2.10) and (2.11)-(2.12) which are coupled with (2.7)-(2.8).

3. METHOD OF SOLUTION

We adopt the shooting method with Runge-Kutta-Fehlberg 45 scheme to solve the initial value problems in PEST and PEHF cases mentioned in the previous section. The coupled non-linear equations (2.7)-(2.10) in PEST case are transformed in to a system of seven first order ordinary differential equations as follows.

$$\frac{dy_1}{d\eta}=y_2,$$
$$\frac{dy_2}{d\eta}=y_3,$$
$$\frac{dy_3}{d\eta}=y_4,$$
$$\frac{dy_4}{d\eta}=\frac{\left(y_4+y_1y_3-2y_2^2-k_1^*\left\{3y_2y_4-\frac{3}{2}y_3^2\right\}\right)}{\left(-\frac{k_1^*y_1}{2}\right)},$$
$$\frac{dy_5}{d\eta}=y_6,$$
$$\frac{dy_6}{d\eta}=Pr\left(4y_2y5-y_1y_6-E\,y_3^2\right), \quad (2.13)$$

The corresponding boundary conditions are

$$y_1(0)=0, y_2(0)=1+\alpha^* y_3(0), y_5(0)=1,$$
$$y_2(\infty)=0,\ y_3(\infty)=0, y_6(\infty)=0. \quad (2.14)$$

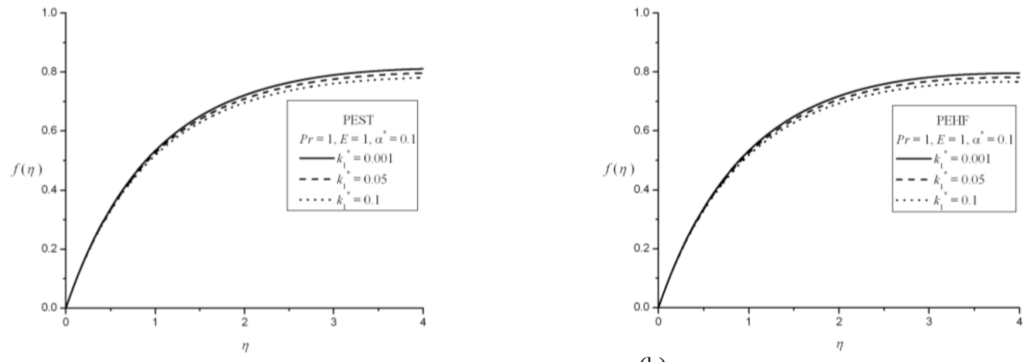
Here, $y_1 = f(\eta)$ and $y_5 = \theta(\eta)$.

Aforementioned boundary value problem is converted in to an initial value problem by choosing the values of $y_3(0)$ and $y_6(0)$ appropriately. Resulting initial value problem is integrated using Runge-Kutta-Fehlberg 45 order method. Newton-Raphson method is used to correct the guess values of $y_3(0)$ and $y_6(0)$.

The decision on an appropriate '∞' for the problem depends on the parameter of values chosen. In view of this, for each parameter combination, the appropriate value of '∞' has to be decided. If the solution of two consecutive '∞'s matches to a desired accuracy, then we take this to be the appropriate '∞' for the given set of parameter combination. First we guess the initial values for the unavailable initial values in the boundary value problem. Obviously the chosen guess values are not the most accurate values and we need to refine them. Newton-Raphson method is used for this purpose. We solve the Eq. (2.13) with these initial conditions, using the Runge-Kutta-Fehlberg 45 order method of four slopes and obtain the solution at '∞'. The solution at '∞' does not match with those given in the problem due to the crude choice of unavailable initial values. So, the method is repeated till the solution of two consecutive '∞'s matches to a desired accuracy. Same procedure is adopted to solve (2.7)-(2.8) and (2.11)-(2.12). The results are presented in several graphs.

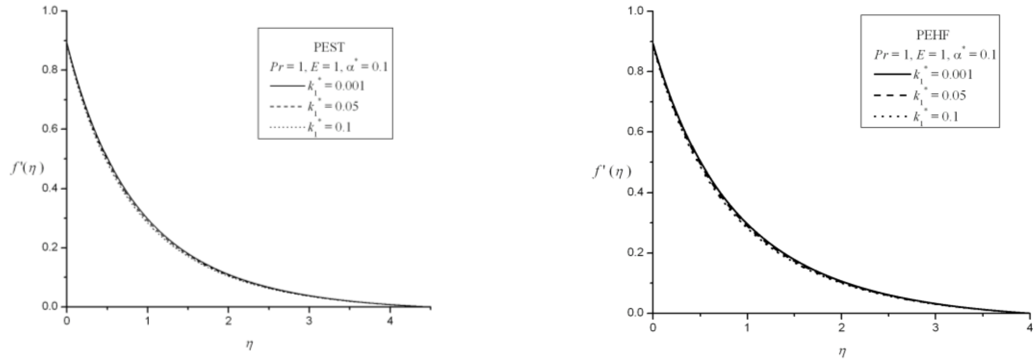
4. RESULTS AND DISCUSSION

The boundary layer flow and heat transfer due to an exponentially stretching sheet in the presence of a viscoelastic liquid is analyzed. The effects of various parameters such as viscoelastic parameter, slip parameter, Prandtl number and Eckert number are shown in several graphs in figures 1 to 8.



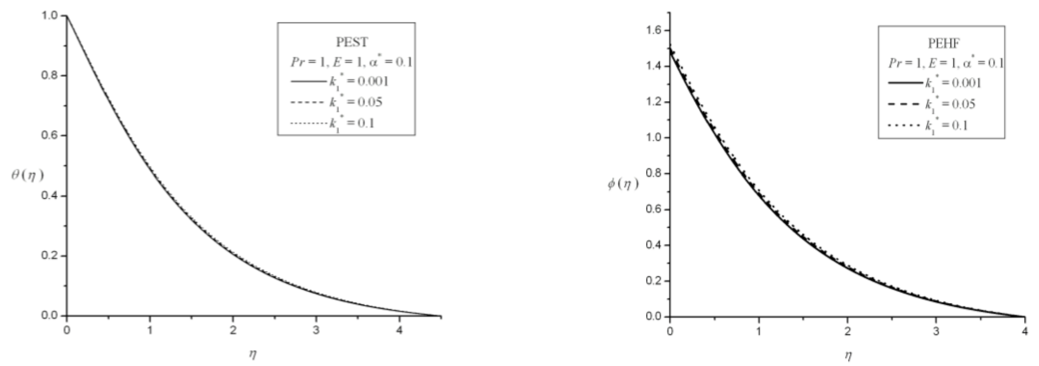
(a) (b)

Fig. 1. Plot of $f(\eta)$ versus η for different values of viscoelastic parameter (k_1^*).



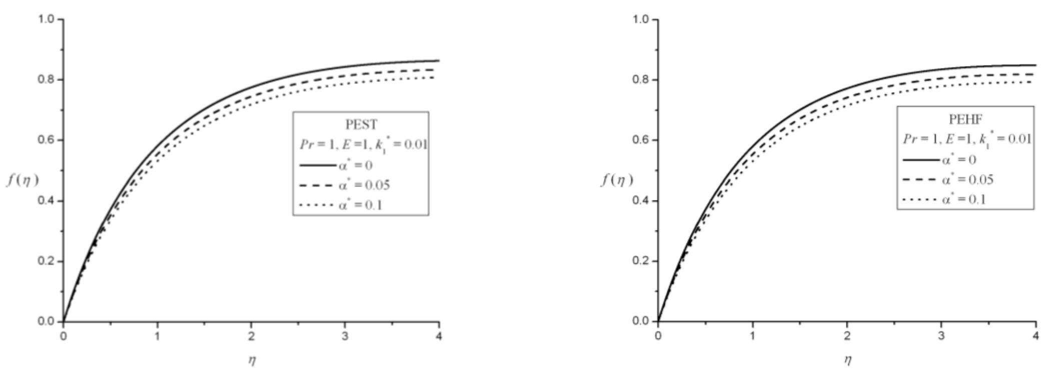
(a) (b)

Fig. 2. Plot of $f'(\eta)$ versus η for different values of k_1^*.



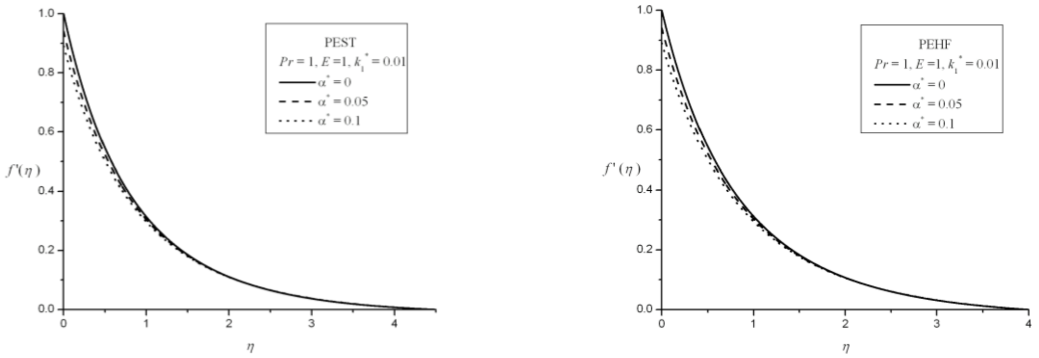
(a) (b)

Fig. 3. Plot of temperature profiles for different values of k_1^*.



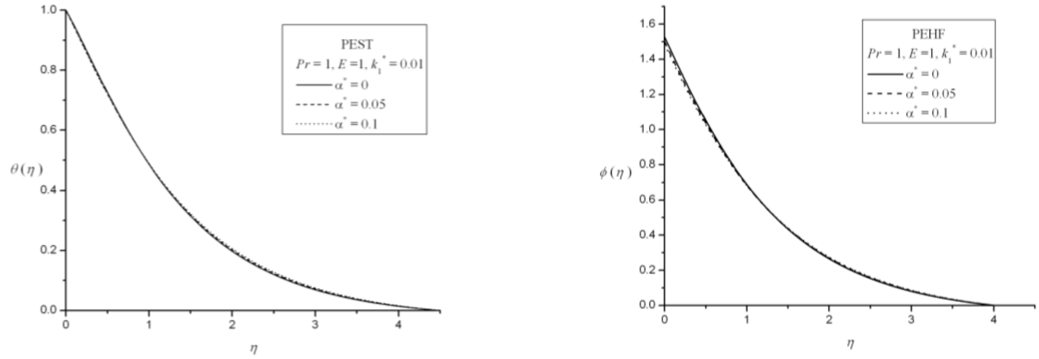
(a) (b)

Fig. 4. Plot of $f(\eta)$ versus η for different values of slip parameter (α^*).



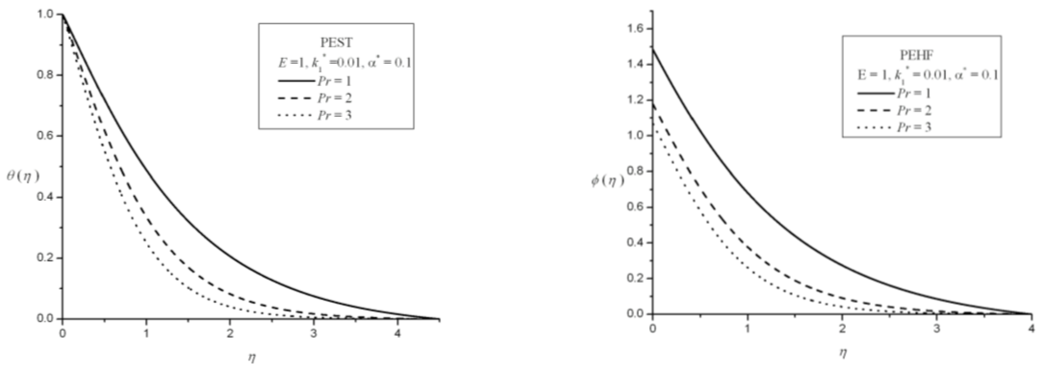
(a) (b)

Fig. 5. Plot of $f'(\eta)$ versus η for different values of α^*.



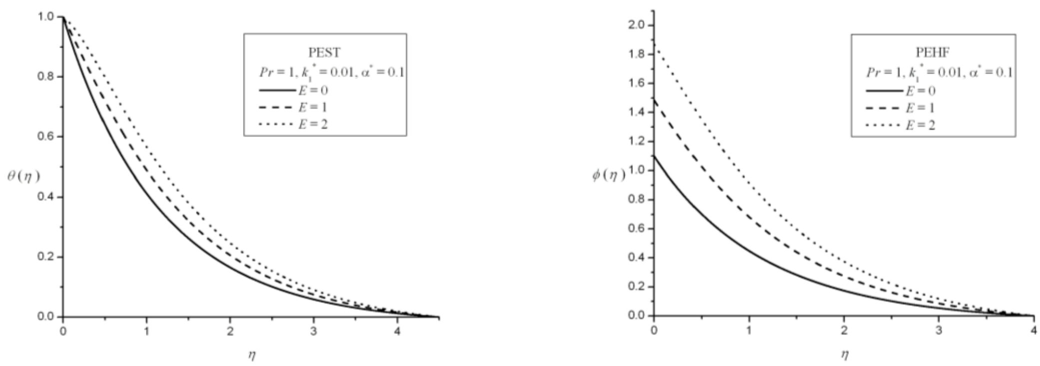
(a) (b)

Fig. 6. Plot of temperature profiles for different values of α^*.



(a) (b)

Fig. 7. Plot of temperature profiles for different values of Prandtl number (*Pr*).



(a) (b)

Fig. 8. Plot of temperature profiles for different values of Eckert number (*E*).

Figure 1– 3 illustrate the effect of viscoelastic parameter k_1^* on the flow and heat transfer in PEST and PEHF cases. It is observed from these plots that $f(\eta)$ and $f'(\eta)$ decrease with increasing values of k_1^*, where as $\theta(\eta)$ increases with increasing values of k_1^*. This means that the increasing values of k_1^* results in thinning of momentum boundary layer and thickening of the momentum boundary layer.

Figure 4 – 6 demonstrates the effect of α^* on the flow and heat transfer. The effect of α^* is similar to that of k_1^* in both PEST and PEHF cases.

The impact of Prandtl number *Pr* on the heat transfer is depicted in figure 7. We infer from the figures that the increase of Prandtl number results in the decrease of temperature distribution at a particular point of the flow region. This is due to the decrease in the thickness of the thermal boundary layer with increasing values of Prandtl number. The increase of Prandtl number means slow rate of thermal diffusion. It is obvious that the wall temperature distribution is at unity on the wall in PEST case for all values of *Pr*, *E* and k_1^*. However, it may be other than the unity in PEHF case due to adiabatic temperature boundary condition.

Figure 8 demonstrates the variation of the temperature profile for different values of Eckert number *E*. The effect of increasing values of *E* is to enhance the temperature in the flow region. This is due to the fact that the heat energy is stored in the liquid considered due to frictional heating.

In order to validate our results, we have compared the values of skin friction coefficient in the absence of viscoelastic parameter and slip parameter with the results of Elbashbeshy (2001) and found them to be in good agreement (Table 1).

Table 1 Comparison of values of skin friction $-f''(0)$ with $k_1^* = \alpha^* = 0$.

$-f''(0)$	
Elbashbeshy (2001)	**Present study**
1.28181	1.281816
1.37889	1.378894
1.4839	1.484389
1.59824	1.598242

Table 2 represents the variations in the magnitudes of the non-dimensional surface velocity gradient $-f''(0)$, wall temperature gradient $-\theta'(0)$ and the wall temperature $\phi(0)$ due to the changes in the numerical values of viscoelastic parameter k_1^*, slip parameter α^*, Prandtl number *Pr* and Eckert number *E*. It can be seen from the table that the magnitude of $-f''(0)$ increases with increasing values of k_1^*. The opposite behavior is seen for the increasing values of α^*. Since the flow problem is uncoupled with the thermal problem, changes in the values of *Pr* and *E* will not affect the value of $-f''(0)$.

Table 2 Values of skin friction $-f''(0)$, wall temperature gradient $-\theta'(0)$ and wall temperature $\phi(0)$ for different values of k_1^*, α^*, *Pr* and *E*.

Parameters		PST		PHF	
		$-f''(0)$	$-\theta'(0)$	$-f''(0)$	$\phi(0)$
		$Pr = 1, E = 1, \alpha^* = 0.1$			
k_1^*	0.001	1.082184	0.559132	1.083152	1.480619
	0.05	1.111439	0.543863	1.112246	1.500746
	0.1	1.145078	0.526923	1.145730	1.523487
		$Pr = 1, E = 1, k_1^* = 0.01$			
α^*	0	1.292758	0.491792	1.293798	1.525866
	0.05	1.179797	0.529716	1.180782	1.500578
	0.1	1.087318	0.556417	1.088255	1.484173
		$E = 1, k_1^* = 0.01, \alpha^* = 0.1$			
Pr	1	1.087318	0.556416	1.08826	1.484173
	2		0.756423		1.177892
	3		0.884869		1.068004
		$Pr = 1, k_1^* = 0.01, \alpha^* = 0.1$			
E	0	1.087318	0.906434	1.08826	1.098930
			0.556417		1.484173
	2		0.206399		1.869416

We notice that the temperature gradient decreases with increasing values of k_1^* and *E* in PEST case and increases with increasing values of k_1^* and *E* in PEHF case. The table also reveals that the effect of increasing values of α^* and *Pr* is to increase $-\theta'(0)$ in PEST case and to decrease $\phi(0)$ in PEHF case.

5. CONCLUSIONS

1. The effect of viscoelastic parameter is to decrease the velocity distribution and increase temperature distribution in the boundary layer.
2. The flow slows down at distances close to the sheet for increasing values of the slip parameter.
3. The effect of Prandtl number is to decrease the magnitude of heat transfer.
4. The energy dissipation due to viscous dissipation has the effect to thicken the thermal boundary layer, increase the temperature profile and hence the heat transfer rate from the surface.
5. The magnitude of the non-dimensional surface velocity gradient decreases with increasing values of viscoelastic parameter.

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