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A Proposed Solution for the Settling Velocity of Coplanar Aggregates of Identical Spheres in Creeping Motion.

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ABSTRACT

This paper derives a solution for the settling velocity of isolated coplanar aggregates of *n* identical spherical particles in creeping motion based on the Wall Shear-Pressure Gradient-Expansion model (WPE) previously derived by the author. The solution is reached by developing geometry-equilibrium mathematical constructs to decide on the dimensions of the free ambient expansions surrounding the aggregates to compute the entire velocity profile. A rational to decide on the orientation and stability of rotation is also proposed. The results compare well with the outcome of available theory and some relevant data sets.

Keywords: Doublet; Triplet; Settling velocity; Wall shear; Expansion; Non spherical particles; Spherical aggregates; Pressure gradient.

NOMENCLATURE

e	tributary ratio
e_{max}	maximum tributary ratio
e_n	maximum tributary ratio for a sphere in an aggregate having *n* units
G_s	specific gravity of solid
G_f	specific gravity of fluid
n	number of spherical solid units in an aggregate
g	acceleration due to gravity
P_f	potential pressure gradient
r	radius of a sphere
r_s	radius of a solid sphere
R	Radius of a spherical expansion
R_n	Radius of a spherical expansion for a sphere in an aggregate having *n* units
u	velocity
V_s	settling velocity of a sphere
V_{sn}	settling velocity of an aggregate of *n* units
φ	mass expansion rate per unit velocity gradient
ρ_f	density of the fluid
ρ_s	density of the solids
μ	viscosity
ξ	spherical expansion
ξ_{max}	maximum spherical expansion
ξ_n	maximum spherical expansion for a sphere in an aggregate having *n* units
ζ_n	ratio of the velocity of an aggregate having *n* units with respect to the singlet
τ_w	wall shear

1. INTRODUCTION

The need for a rational approach to address problems in low Reynolds hydrodynamics has arisen throughout centuries in science and engineering. Particularly, the problem of settling particles, affect our daily lives directly through, climate, potable water, waste water treatment, food processing, biotechnology, etc. Although, a glimpse to our daily lives highlights the importance of reaching a good understanding of settling phenomena the developments have been rather slow. Settling phenomena has been research widely in sedimentation, chemistry, colloidal science, biotechnology and hydrodynamics amongst other fields with a rather modest improvement. The highest end of our understanding is known as Stokes' law after the novel work of Gabriel Stokes (1851). Stokes' law does not need to be introduced; it is widely known and its derivation is readily available. Stokes' law has been since, the corner stone of further research, to reach, 75 years later, a solution for the doublet by Stimson and Jeffery (1926). An "exact" solution for the triplet has not been reached.

As noted in Johnson C. P. et al (1996), the trend of previous research to derive the estimation of the settling velocity of aggregates, has been the assumption that Stokes' law is applicable. The relationships are then accomplished by inclusion of a fractal or equivalent dimension with additional terms to account for the influence of porosity, permeability, density and mass. Despite of these research efforts substantial deviations with respect to the experimental results are not uncommon.

The fundamental assumption behind the derivations in this work differs from the majority of previous research. This paper does not follow, Stokes' law; it follows the Wall shear – Pressure gradient – Expansion Model (WPE) proposed in Mendez, Y. (2011, 2012 and 2014). The reader can refer to Johnson C. P. et al (1996) for further discussion and research references for the application of Stoke's law based relationships to derive the settling velocity of aggregates.

This paper is not concerned with the processes promoting the formation of aggregates and is only related to the fluid mechanisms controlling the velocity of aggregates of solid spheres in creeping motion.

2. DATA SETS AND CORRELATIONS

This paper proposes a solution for the settling velocity of coplanar isolated aggregates of solid spheres of equal size and density in a liquid Newtonian fluid with viscosity less than *0.01 Pa-s* and velocity gradient, defined as wall shear τ_w divided by μ in the order of $160\ s^{-1}$. Isolated aggregates, in this work, refer to aggregates bonded together by a surface force only, as in DLVO theory, and their corresponding expansion. The expansion limit is at some distance away from any solid surface or neighboring expansion so that the surrounding fluid is quiescent. Ela W. P. et al. (1999) points out the coalescence and the fractal approach as the preferred available methods for modeling the settling behavior of aggregates and some of their limitations; and highlights the lack of experimental systems for testing of different configurations. In addition to the difficulties pointed out by Ela, W. P., the "manufacturing" of spherical aggregates with precise configurations is still a challenging task and the difficulty of biased interpretations where "ambient noise", Mendez, Y. (2012), has not been removed pose additional challenges in the design of experiments and their interpretation. While a number of relevant data sets referenced in Johnson C. P. et al (1996) are available, an accurate characterization of the geometry of individual aggregates was not made on those references and as such impossible to evaluate under this model. In general, it appears that in majority, the problems associated with experimental settings suggest that the observed behavior for a given particle size exceed the actual velocity under ideal conditions. Other issues, however, related to poor characterization of the systems, such as the actual particle size, the exact spherical geometry and the angle and space between particles may have resulted in observed velocities that are less than the ideal portrayed in our models. The latter issues are expected to be rather scarce. Although, we choose to exercise caution in the comparison with experimental data, a velocity estimation of the type "less than observed" is considered a better estimate than "more than observed". Some of the most reliable data sets, Bart, E (1959) and Eveson et al. (1959) for doublets of identical spheres, have been examined previously by Happel and Brenner (1983) which in general indicate that a doublet Falling Normal to their Line of Centres (FN) settles 42% faster than the single sphere and 54% faster for doublets Falling Parallel to their Line of Centres (FP) with some scatter. Although, it is not attempted to model the dynamics of individual particles settling along identical neighboring particles, referred to as Unbound Aggregates (UA) (which appears fairly feasible), experimental results of this nature will be used to assess the relative magnitude of the forces controlling the motion and whether the observations support our model or not. Reliable data sets for triplets with rigorous definitions of dimensions and configuration are unavailable. For reference, we consider measurements of settling velocity for triplets by Ela, W. P. (1999) which were completed in a continuous flow water column and normalized to 20^oC. The experimental conditions include dynamics that appear difficult for interpretation of the settling velocity and will not be examined in detail.

3. BRIEF ABOUT THE WPE

For spheres, velocity profile within a spherical ambient expansion is written as:

$$u = \frac{-P_f}{2\mu}\left(\frac{r^2}{3} + \frac{2R^3}{3r} - R^2\right) \tag{1}$$

where Pf is the potential pressure gradient, μ is the viscosity, u is velocity, r is radius and R is radius of the expansion.

From r = the radius of the solid sphere rs and $r = R$.

The settling velocity Vs computed at the wall of the solid sphere is computed as:

$$V_s = \frac{-P_f r_s^2}{2\mu}\left(1 + \frac{2e_{\max}}{3} - \left(1 + e_{\max}\right)^{\frac{2}{3}}\right) \tag{2}$$

where e_{max} is the tributary ratio defined as the ratio of the volume of the ambient expansion to the volume of the solid sphere and computed as:

$$e_{\max} = \frac{\left(\rho_s - \rho_f\right)g}{P_f} \tag{3}$$

where ρs is density of the solid sphere, ρf is density of the fluid and g is acceleration due to gravity.

From the volumetric relationship between the volume of the ambient expansion to the volume of the solid, R is computed as

$$R = r_s \left(1 + e_{\max}\right)^{\frac{1}{3}} \quad (4)$$

and the potential pressure gradient as

$$\frac{\mu \rho_f}{\varphi} = P_f \quad (5)$$

where φ is the mass expansion rate per unit velocity gradient equal to $1.148 \times 10^{-3} kg\text{-}s/m^2$.

The relationships above seem to break down when the velocity gradient is about $160\ s^{-1}$ and the Reynolds number for all data sets has been less than 1. The kinematic viscosities have been in the range of 1×10^{-6} and $6 \times 10^{-7}\ m^2/s$ and considered to be very important in deciding the applicability of the relationships. High kinematic viscosities and adhesive forces are expected to operate in different dynamics.

Note the spherical expansion ξ between brackets in Eq. 2. ξ is an implication of the volumetric relationships associated with the computation of the velocity by Eq. 1 and is the end result of the fact that the ambient expansion is *emax* times the volume of the tributary solid mass (kg/m^2) of the particle, (and hence *emax* times the volume of the solid sphere) holding the same value for different particle sizes. It is an important device as its form varies with the geometry and varies in magnitude with variations in temperature and relative difference of the density of the solids to the density of the fluid. Whenever used in the context of the WPE we must bear in mind that there is always a velocity profile involved. The expansion is a key insight of the WPE in the derivation of a solution for the spherical aggregates in this paper. We here present a rational construction of the expansion profiles about spherical aggregates based on the basic understanding that its size is prescribed by the relative difference of density of the solids to the density of the fluid, the potential pressure gradient and available space surrounding the aggregates. In essence, we are better considering ourselves computing shear stress and force through expansion dimensions that are calibrated to satisfy an equilibrium condition i. e. that the pressure gradient times the tributary volume (m^3/m^2) in the expansion must be equal to the shear stress (wall shear in N/m^2) of the aggregate. It will be seen that geometric constrains to the free expansion limit the mobilization of stress in areas where the expansions overlap other expansions or solid surfaces within the aggregate.

4. HYPOTHESIS

The doublet consists of two solid spheres of radius *rs* bonded together by a surface force and two overlapping spherical expansions of radius R_2. Note that two perfect spheres bonded together by a surface force that is equally distributed throughout the entire surfaces FN is an aggregate that is Internally Unstable (IU) for moment as the contact surface by itself cannot mobilize a resisting moment at their point of contact. The spheres unavoidably rotate regardless of the magnitude of the surface stress. Two spheres in the envisioned configuration can be stable against rotation if the surface force (stress) is not evenly distributed and/or the "double layer" nature of the surface force allow for the mobilization of a resisting moment and/or the spheres are not perfect and have small irregularities. Although, we consider perfect spherical geometries and expansions, the premise is that they are Internally Stable (IS) as there is always a resisting moment of sufficient magnitude at their contact point by any of the means envisioned. Visualizing the system we note:

- The wall shear of each particle can be computed.
- The two spherical expansions overlap.
- The assumption of internal stability prevents rotation.

We draw the following preliminary conclusions:

- In the portion of the expansion defined by a cone which base is the intersection of the spheres and the apex is the center of the solid sphere, the fluid and the dynamics are not free to expand; and the mobilization of shear stress is limited by the limited space. In further discussion, this configuration will be quoted as the overlapping expansion and the reminder of the expansion will be quoted as the free expansion.
- The transfer of the submerged weight of the particle to the fluid by a single value of wall shear is not feasible due to the physical constrain of overlapping expansions.

We formulate the following hypothesis for the dynamics of mobilization of the driving force by the fluid and the subsequent rational to compute the velocity based on understanding of the WPE:

To satisfy equilibrium, the expansion fluid missing in the overlapping expansions is distributed to enlarge the free expansions. As such, the conclusion drawn from the dynamics, that the expansion fluid needs to be e_{max} times the volume of the spheres (to be able to mobilize the submerged weight), provides knowledge of the volume; the solution can thus be obtained by characterizing the dimensions of the free expansion to satisfy this requirement and compute the corresponding shear stress and velocity of the aggregate through the enlarged expansions. The discussion below is intended to test this hypothesis.

It is important to note that the dimensions of the expansion are extremely important in reaching an accurate computation of settling velocity. As such the efforts made in this paper to characterize the geometry accurately are a key aspect of its purpose.

5. DOUBLET

The notation *Rn*, with *n* equal the number of aggregates, will be used to denote the radius of the spherical expansion computed about a unit of the aggregate containing *n* units. Geometry wise the two overlapping expansions are two spherical caps whose volume V_2 can be computed exactly as:

$$V_2 = 2\left(\frac{1}{3}\pi h^2\left(3R_2 - h\right)\right) \tag{6}$$

Where h is the distance between the spherical surface of the expansion and the plane tangent to the spheres at their point of contact measured on a line joining the centers. Setting $h = (R_2+rs)$ we obtain:

$$V_2 = 2\left(\frac{1}{3}\pi\left(R_2 + r_s\right)^2\left(2R_2 - r_s\right)\right) \tag{7}$$

We continue by noting that *2(4/3πrs3)(1+emax)* computes the volume of the system containing the two spheres and expansions and two times the right side of Eq. 7 computes the same. Eqs. 5 and 3 provide the means to solve for e_{max} and compute this volume. We define e_2 as the maximum tributary volume of the doublet to satisfy $R_2=rs(1+e_2)^{1/3}$ to write our equality as,

$$2\left(\frac{4}{3}\pi r_s^3\left(1+e_{\max}\right)\right) = 2\left(\frac{1}{3}\pi\left(r_s\left(1+e_2\right)^{1/3} + r_s\right)^2\left(2r_s\left(1+e_2\right)^{1/3} - r_s\right)\right) \tag{8}$$

This simply equates Eq. 7 to the prescribed volume of the system. The left side of Eq. 8 is the volume without the geometry and the right side the same volume for the given geometry. After the mathematical reshuffling we obtain,

$$e_{\max} = \frac{1}{4}\left(2\left(1+e_2\right) + 3\left(1+e_2\right)^{2/3} - 1\right) - 1 \tag{9}$$

We need to clarify e_2. The tributary volume is defined as the volume of the spherical system about a unit VR_2 (or volume of the overlapping sphere) minus the volume of the solid sphere divided by the volume of the sphere. This quantity is a rigorous definition of the WPE resulting from the dynamics and the characterization of the wall shear as the driving force. It should be noted that the entire sphere defined by e_2 from Eq. 9 and R_2 retains a greater volume than $4/3\pi rs^3(1+emax)$. For doublets e_2 is to define the volumetric relationship within a sphere of radius R_2 whose truncated expansion retains a volume of $4/3\pi rs^3(1+emax)$. In general the notation e_n, with n equal the number of spherical units within an aggregate, will be used to denote the volume of the n ambient truncated expansions divided by the volume of the n solid spheres. Continuing our discussion, solving for e_2 in Eq. 9 the spherical expansion ξ_2 for the doublet becomes available as:

$$\xi_2 = \left(1 + \frac{2e_2}{3} - \left(1+e_2\right)^{\frac{2}{3}}\right) \tag{10}$$

The velocity of the doublet Vs_2 follows as computed across the corresponding truncated spherical expansion as:

$$V_{s2} = \frac{-P_f r_s^2}{2\mu}\left(1 + \frac{2e_2}{3} - \left(1+e_2\right)^{\frac{2}{3}}\right) \tag{11}$$

, and the ratio of the velocity of the doublet with respect to the singlet of identical particles ζ_2 is defined by the ratio of their corresponding expansions ξn (between brackets in Eq. 10). It is emphasized that Eq. 9 is the mathematical construct allowing the construction of the expansion dimensions to satisfy equilibrium and geometry, Eq. 11 with the corresponding subscript n, as in e_n is the end result of applying the dynamics in the expanded expansion to compute the velocity at the wall of the solid sphere in an aggregate having n units and Eq. 1 with the subscripts n, as in Rn to denote the number of spheres forming the aggregate can be used to compute the velocity profile. The velocity profile can thus be computed from $r = Rn$ to $r = rs$, where rs is the radius of the solid sphere. All these relationships are the result of a sound derivation from the dynamics and geometry. Hence, the goal for the computation of the velocity for all aggregates in this paper is to find the mathematical constructs for defining the size of the enlarged expansions (as portrayed by e_n) as they correspond to the number of units in the aggregates, the geometry and the volumetric relationships satisfying equilibrium.

Noting that φ have been determined to be approximately *1.148 x 10-3 (kg-s)/m2*, we find, the pressure gradient for our fluid properties as *(μρf)/φ = 871 Pa/m* and the maximum tributary ratio *emax* = $g(\rho s-\rho_f)/P_f$; where g is the acceleration due to gravity, is computed as 18.57 for our solid properties (Gs = 2.65). After solving for e_2 in Eq. 9, the maximum spherical expansion for the singlet is *6.126* and *9.07* for the doublet, which indicate a velocity *48%* greater for the doublet. As ζ_2 and in general ζn depend on the relative size of the expansions, its value is not a constant for variations in temperature and fluid. Note that Stimson and Jeffery (1926) seem to have concluded that the velocity of the doublet is a single factor greater than the singlet for any doublet, a fact that appears very unlikely given the variations of the relative size of the expansions with temperature. Both experimental results by Ela, W.P. et al. (1999) and Bart, E. (1959) for the case of IS doublets FN indicate *40%*, which correspond to a specific gravity of solids *Gs* of about *1.8* for our fluid properties. Other data sets examined by Eveson, G. F. (1959) are for IU doublets and Unbound Aggregates (UA). UA to denote groups of solid units settling at close distances between each other but not touching so that their motion is influence by neighboring overlapping fluid expansions. Velocity wise, the latter condition is out of the scope of this work, as there is no attempt for deriving the velocity for this condition, shear stress wise or forces wise the need to examine results of this nature will arise in further discussion in this paper.

6. STABILITY OF ROTATION AND ORIENTATION

When one envisions a settling aggregate, the immediate question that arises is, what is the orientation of the particle as it settles?. The immediate conclusion of equilibrium is that for all conditions, IS aggregates or singlets rotate, except, when the summation of moments about the center of mass of the system is equal to cero (ΣMc = 0). In this discussion we refer to an IS settling aggregates as Externally Stable (ES) or Externally Unstable (EU) when equilibrium considerations indicate that they do not rotate $\Sigma Mc = 0$ or rotate $\Sigma Mc \neq 0$ respectively as it settles. As such, notation of the type ISEU for Internally Stable Externally Unstable apply for aggregates with units in fixed positions with respect to each other and rotate overall as they settle.

For assessment of equilibrium we need to reach an understanding of the forces and stresses acting on the particle. For this purpose we examine data sets and observations of UA on the grounds of the WPE to determine the relative magnitude of the forces acting on them. The understanding reached so far indicate:

A) We can compute the shear stress across a free expansion of given dimensions.

B) The shear stress is reduced whenever the free condition is breached. It is inferred by equilibrium conclusions that the total force mobilized across an overlapping expansion can be estimated as the product $P_f Vo$; where Vo is the volume of fluid within the overlapping expansions.

Jayaweera K . 0. L. F. et al. (1964) presented in a straight forward manner settling experiments of IUSs as follows:

1. Equal-sized spheres falling side-by-side
2. Equal-sized spheres falling vertically one behind the other
3. Two equal spheres with line of centres inclined to horizontal
4. Equal-sized spheres released in a horizontal straight line
5. Clusters of 3 to 6 equal spheres

We state “a singlet surrounded by a free expansion is IS”. Observations 1 to 5 indicate that the spheres do not rotate once they reach a certain distance from the cluster.

Statement A and B hold true. The top of doublets always rotate inward towards each other as the shear stress is less in the overlapping expansion. In addition, this effect continues at some distance. The center of mass of the particle tends to take a position below the higher shear stress (or rotate about it) and the particles thus separate as they settle until the entire expansions become free. Moreover, in a horizontally aligned triplet, the particle towards the center always advance ahead of the other two because the double overlapping expansions reduce the friction as compare to a single overlapping expansions for the rest. Friction to denote in general, shear stress that is not being computed whose relative magnitude can be assessed by geometric considerations.

We state “the doublet at Re (Reynolds number) <0.01is IS and ES”. This doublet, released side by side did not separate or rotate. The internal stability is presumed to be related to cohesive properties of the fluids employed. At the smaller Re the cohesive forces in the fluid attached to the small spheres become significant. The external stability will be addressed in fore coming discussion.

In addition we note:

C) A reduction in friction occurs at the location of the wake which appears to extend to a greater distance than the expansion. In all experiments, a sphere that is initially above another sphere accelerates to touch the leader and then slides round the leader to take a stable position on a horizontal plane. Then rotate and separate as noted previously.

We propose a simple rational to decide on the stability of rotation and the orientation of IS aggregates under the scope of this paper based on A, B and C. We state “an IS doublet FP is EU”. We can see that the shear stress acting on its frontal face and the reduced friction towards the back due to A, B and C can be presumed to induce a pair about the centre of mass. We state “an IS doublet FN is ES”. Again, based on A, B and C we can presume that the pair about the center of mass is cero. This is also supported by the IS doublet at Re<0.01 which was observed to settle side by side without rotation.

Equation 11 has been derived to account for the two identical free expansions developing freely across the fluid which appear to match the conditions of the IS doublet FN. This draws, in general, the conclusions regarding the computation accomplished by Eq. 11.

7. TRIPLET

We consider a triplet in its closest possible arrangement as shown in Fig. 1 (a). Planes JA, JB and JM define the volumes that “belong” exclusively to each sphere. The volume defined by the three expansions of radius R_3 is simply three overlapping spheres. According to the rational provided for the doublet, we are able to compute the velocity Vs_3 of the triplet across the profile of the truncated expansion (b).

The volume and area of the three overlapping spheres is of great interest in molecular diffusion and other problems and not a trivial matter. We embrace the problem as follows: On Fig. 1(b) and (c), our goal is to subtract the volume of the system VSY enclosed by the segments AX and XB and the cord AB of the sphere of radius R_3 with centre in O to the volume of the entire overlapping sphere VR_3 of radius R_3, as shown in fig. 1(b) and (c). AX and XB are tangent to the solid sphere of radius rs. One of the caps (enclosed by the segments BD and the cord BD) of volume VC, which can be computed exactly as $1/3\pi H^2(3R\text{-}H)$, where H is the length FG, is envisioned as the rotation of the other by the angle of rotation θ. For spheres in their closest arrangement θ can be verified to be 60º. It can be seen that the summation of the volume of the two caps is equal to the volume VSY enclosed by the segments AX, BX and the cord AB plus one wedge

XCD of volume *Vw* defining the intersection of the caps. We propose the following relationship for the volume of the wedge V_W:

$$V_w = \frac{1}{3}\pi\left(\frac{h^6}{H^5}\right)^2\left(3R - \frac{h^6}{H^5}\right) \tag{12}$$

Where *h* is the length XE and H the segment FG defined previously. Equation 12 is in a sense, an expansion relationship of the cap of volume VC whose dimension can be defined entirely by R_3, *rs* and θ. We will see how the exponents are justified in fore coming discussion. The V_W can be seen to turn to VC when θ is 0 and 0 when θ is equal to the aperture angle of the cap α defined by the segments OB and OD. The segment OX can be computed as $rs/\cos(\theta/2)$ and $R_3 = rs(1+e_3)^{1/3}$, hence $XE=h=R_3-OX$ or $h=rs((1+e_3)^{1/3}-1/\cos(\theta/2))$. By definition $H= rs((1+e_3)^{1/3}-1)$. Let us write Eq. 12 as per the above definitions:

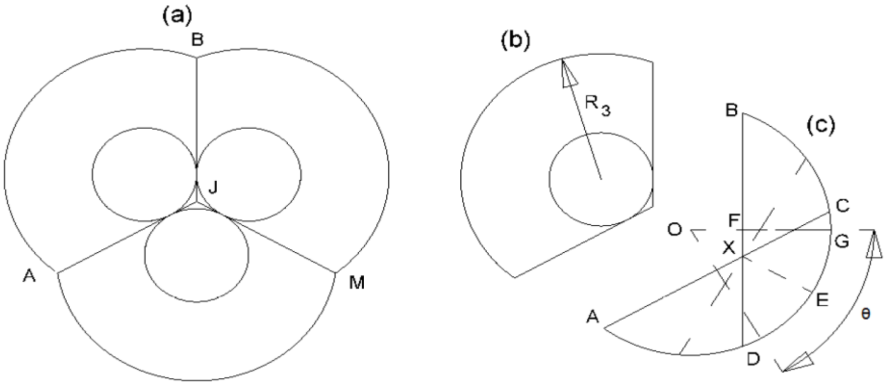

Fig. 1. Geometry of the triplet and its expansion.

$$V_W = \frac{1}{3}\pi\left(\frac{\left(r_s\left((1+e_3)^{1/3} - \frac{1}{\cos(\theta/2)}\right)\right)^6}{\left(r_s\left((1+e_3)^{1/3}-1\right)\right)^5}\right)^2\left(3r_s(1+e_3)^{1/3} - \frac{\left(r_s\left((1+e_3)^{1/3} - \frac{1}{\cos(\theta/2)}\right)\right)^6}{\left(r_s\left((1+e_3)^{1/3}-1\right)\right)^5}\right) \tag{13}$$

VC for the cap of height H takes the form,

$$V_C = \frac{1}{3}\pi\left(r_s\left((1+e_3)^{1/3}-1\right)\right)^2\left(r_s\left(2(1+e_3)^{1/3}+1\right)\right) \tag{14}$$

The volume VSY then becomes available as VSY=2VC-Vw and the volume of the truncated sphere VTS in Fig. 1 (b) in a unit of the triplet equal the volume of the sphere VR_3 (computed as $4/3\pi rs^3(1+e_3)$) minus VSY. Then 3 times the volume of the truncated sphere equal the volume of the system of the triplet V_3 as,

$$V_3 = 3\left(\frac{1}{3}\pi r_s^3\left(\begin{array}{l}4(1+e_3) - 2\left((1+e_3)^{1/3}-1\right)^2\left(2(1+e_3)^{1/3}+1\right) + \\ \frac{\left((1+e_3)^{1/3} - \frac{1}{\cos(\theta/2)}\right)^{12}}{\left((1+e_3)^{1/3}-1\right)^{10}}\left(3(1+e_3)^{1/3} - \frac{\left((1+e_3)^{1/3} - \frac{1}{\cos(\theta/2)}\right)^6}{\left((1+e_3)^{1/3}-1\right)^5}\right)\end{array}\right)\right) \tag{15}$$

Then V3 must accommodate the prescribed volume of $3(4/3\pi rs^3(1+emax))$. Further reshuffling leads to,

$$e_{\max} = \frac{1}{4}\left(\begin{array}{l}4(1+e_3) - 2\left((1+e_3)^{1/3}-1\right)^2\left(2(1+e_3)^{1/3}+1\right) + \\ \frac{\left((1+e_3)^{1/3} - \frac{1}{\cos(\theta/2)}\right)^{12}}{\left((1+e_3)^{1/3}-1\right)^{10}}\left(3(1+e_3)^{1/3} - \frac{\left((1+e_3)^{1/3} - \frac{1}{\cos(\theta/2)}\right)^6}{\left((1+e_3)^{1/3}-1\right)^5}\right)\end{array}\right) - 1 \tag{16}$$

leads to the velocity and $\zeta_3=2.03$. Equation 16 is the mathematical construct to define the geometry of the expansion profile for the triplet which can be computed from $r = rs$ to $r = R_3$.

Using the rational provided above, it can be seen that the volume of the expansions for the aligned triplet V_{3A} can be computed as 3VR-4VC. As per definitions we reach,

$$V_{3A} = \frac{4}{3}\pi r_s^3 \left(\frac{3(1+e_3) - \left((1+e_3)^{1/3} - 1 \right)^2}{\left(2(1+e_3)^{1/3} + 1 \right)} \right) \quad (17)$$

Then, 3 times the prescribed volume of the triplet equal V_{3A}. The end result is written as,

$$e_{\max} = \frac{1}{3} \left(\frac{(1+e_3) - \frac{1}{2}\left((1+e_3)^{1/3} - 1 \right)^2}{\left(2(1+e_3)^{1/3} + 1 \right)} \right) - 1 \quad (18)$$

The velocity becomes available and $\zeta_{3A}=1.779$. The relationship is the same for all triplets for which the truncated expansions of 2 units are not touching.

Equation 12 appears applicable for must cases, when the triplet is its closest arrangement. It can be seen by geometry that the formation of a wedge only occurs when $R_3 > rs/cos(\theta/2)$ or $e_3 = (1/(cos^3(\theta/2))-1$or $e_3 = 0.54$. The exception occurs for the latter condition in which we can see that $V_3(e_3<0.54) = 3VR_3\text{-}6VC$. The relationships can be easily derived but not presented because they cover cases for specific gravity of solids exceeding the specific gravity of fluids by less than about 4%.

With regard to the stability of rotation and orientation of the triplet we note that the triplet always define a plane. The terms FN and FP will be used in analogy with the doublet but in reference to the plane and not the line. We state, "an IS aligned triplet and a IS triplet in its closest arrangement FN are ES". The equilibrium condition $\Sigma Mc = 0$ is satisfied by consideration of A and B. In addition we state "the same IS triplets FP are EU". $\Sigma Mc \neq 0$ as indicated for observations 1 to 5 by means of C and A (the stress on the frontal face), for clusters up to $n = 6$, the reduced friction on the back of the sphere yield for an acceleration of the particles following the leader to draw level on a symmetric arrangement. The computations accomplished by the equations above are for free expansions and are expected to model the conditions of the ISES aggregates.

8. COMPARISON WITH AVAILABLE THEORY AND DATASETS

As per available theory from Stimson and Jeffery (1926) the doublet is expected to settle by a factor of *1.42* faster than the singlet. This value compares well with some experimental data from Bart, E (1959) and Eveson et al. (1959) with significant scatter. The computed expansion ratio ζ_2 for specific gravity of solids of *2.65* in water at 20^o *C* is *1.48* using this model. Note that the expansion ratio is not constant and as such, ζ_2 varies with temperature and fluid. For example, using the same properties except temperature at 15^o *C*, ζ_2 can be verified to be *1.46*. Also, using the relationships controlling the expansion size and pressure gradient, a specific gravity of solids of about *2.1* in water at 20^o *C* would have yielded the factor derived by these authors.

Ela, W. P. et al. (1999) measured $\zeta_3 \approx 1.5$ for their experimental conditions (in a continuous flow water column and normalized to 20°C) and computed coalesced $\zeta_3 \approx 2.07$. ζ_3 computed using this model delivered 2.03 for the same solid and fluid properties of the doublet. The same remarks above for the doublet apply for the triplet with regard to the expansion ratio.

The hypothesis behind these derivations appears reasonably justified by the evidence.

9. MORE THAN THREE SPHERES

The consideration in this section is for spheres in their closest arrangement as shown in the coplanar aggregate of 35 spheres shown in Fig 2. By adding spheres to the triplet on the same plane we note:

- a hexagonal honeycomb like structure forms by joining lines tangent to the spheres at their point of contact,
- the lines define the truncated expansions that "belong exclusively" to each sphere when we extend them in a "radial" direction to the limits of the expansion,
- n aggregates define *L* number of lines and *S* number of vertices within the honeycomb arrangement.

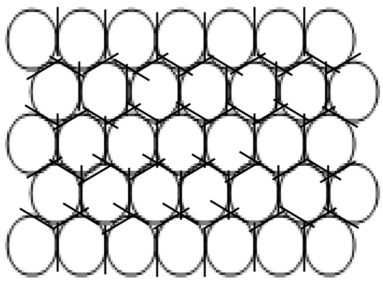

Fig. 2. Coplanar aggregate of 35 spheres

It can be seen in Fig. 2 that the geometry at each vertex is the same geometry studied for the triplet. i.e. when we remove six caps from the three spheres we are in addition subtracting three additional wedges, as every pair of caps that overlap duplicate a wedge. The volume of the plane aggregate of *n* spheres *Vn* is defined as $Vn = n(VRn) - 2L(VC) + 3S(V_W)$. Where *VRn* has been defined as the volume of one spherical system (without removing the overlapping caps) in an aggregate containing n units. According to the WPE the requirement for equilibrium of forces dictate that the volume of the aggregate *Vn* is to satisfy $n(4/3\pi rs^3(1+emax)) = Vn$. Fig. 2 shows a plane aggregate of 35 spheres for which $L = 82$ and $S = 48$.

Substituting our definitions for *Vn* we obtain ↓

$$V_n = \frac{1}{3}\pi r_s^3 \left(\begin{array}{l} n4(1+e_3) - 2L\left((1+e_3)^{1/3} - 1\right)^2 \left(2(1+e_3)^{1/3} + 1\right) + \\ 3S \dfrac{\left((1+e_3)^{1/3} - {1}/{\cos(\theta/2)}\right)^{12}}{\left((1+e_3)^{1/3} - 1\right)^{10}} \left(3(1+e_3)^{1/3} - \dfrac{\left((1+e_3)^{1/3} - {1}/{\cos(\theta/2)}\right)^6}{\left((1+e_3)^{1/3} - 1\right)^5} \right) \end{array} \right) \tag{19}$$

and e_{max} is thus

$$e_{\max} = \frac{1}{4} \left(\begin{array}{l} 4(1+e_3) - 2\dfrac{L}{n}\left((1+e_3)^{1/3} - 1\right)^2 \left(2(1+e_3)^{1/3} + 1\right) + \\ \dfrac{3S}{n} \dfrac{\left((1+e_3)^{1/3} - {1}/{\cos(\theta/2)}\right)^{12}}{\left((1+e_3)^{1/3} - 1\right)^{10}} \left(3(1+e_3)^{1/3} - \dfrac{\left((1+e_3)^{1/3} - {1}/{\cos(\theta/2)}\right)^6}{\left((1+e_3)^{1/3} - 1\right)^5} \right) \end{array} \right) - 1 \tag{20}$$

an aggregate of 8 spheres, $L = 14$ and $S = 7$ yielding $\zeta_8=3.51$ and for Fig. 2, $\zeta_{35}=4.97$. Note that due to the inexact calculation of the volume of the wedge Eq. 20 is to revert into the complex numbers to find a solution when n increases to a certain number. One thing we know by simple geometric considerations is that when n tends to infinite, the volume of each "hexagonal spherical expansion" within Fig. 2 will tend to be $4/3\pi rs^3(1+e_{max})$ thus reaching a stable quantity and velocity as for a slender plate described by Mendez Y. (2014). Also, note that the ratios L/n and S/n will tend to 3 and 2 respectively at infinite turning the coefficients of the second and third term into 6. The exactness of the computation of the volume of the wedge can hence be refined by finding the greatest integer exponent p on (h^p/H^{p-1}) that satisfy Eq. 20 with the coefficient 6 at infinite for the second and third term. With the exponent $p = 6$ we reach $e_\infty = 81.46$ for $e_{max} = 18.57$ and $\zeta_\infty=5.94$. For exponents less than 6 the relationship starts reverting slowly towards the complex numbers and for greater exponent it reverts upon reaching the exponent 7. We have inferred that the maximum accuracy is obtained from the exponent 6. Note the difference between ζ_∞ *(5.94)* and ζ_{35} *(4.97)*. Although in natural or experimental conditions the coplanar configuration should be extremely rare, it can be seen that adding units to the arrangement does not further limit the available space in the truncated expansion to allow for the fluid response individually. The equations developed for the plane condition should not be taken as to suggest that the increase in velocities for three dimensional arrangements of n units is as low as those suggested by the coplanar condition. This can be seen when we note that the volume density of fluid η defined as the volume of fluid and the volume of solids within an aggregate (commonly quoted as porosity) can be as low as 0.5 or less, as opposed to the tributary volume in the order of 18 to allow for the fluid response. The limited space hence mandate the mobilization of force through the "shell" of the aggregate and the increase of velocity with the n units is much greater than for the plane condition. The relationships for the three dimensional condition are being studied and will be the subject of a future article.

Finally, the same considerations apply for the stability of rotation and orientation of the coplanar aggregates in its closest arrangement so that we conclude that the IS coplanar aggregate of n units FN is a ES and Eq. 11 with the corresponding subscripts is deemed to accomplish the computation for the velocity under this orientation.

The relationships for the aligned aggregate of n units can be easily developed by similar geometric considerations and there is no need to present them.

10. CONCLUSION

The settling velocity of particles and aggregates is a poorly understood subject. The model presented has been envisioned to satisfy equilibrium considerations under fluid dynamics and appears in fair agreement with available experimental data and theory. Although, a rational solution for the problem of the triplet has appeared for many years as an overwhelmingly difficult task the WPE offers a simple rational to embrace an intuitively plausible solution for the problem, easy to follow by non specialists.

The geometries associated with aggregates entrain problems that have not been solved by geometry. The use of computer software, however have decreased this difficult problem and the application of the WPE in combination with software appear to have great potential for the solution of practical problems and additional research needed in this field.

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