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The Mixed Finite Element Multigrid Preconditioned MINRES Method for Stokes Equations

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ABSTRACT

The study considers the saddle point problem arising from the mixed finite element discretization of the steady state Stokes equations. The saddle point problem is an indefinite system of linear equations, a feature that degrades the performance of any iterative solver. The heart of the study is the construction of fast, robust and effective iterative solution methods for such systems. Specific attention is given to the preconditioned MINRES solver PMINRES which is carefully treated for the solution of the Stokes equations. The study concentrates on the block preconditioner applied to the MINRES to effectively solve the whole coupled system. We combine iterative techniques with the MINRES as preconditioner approximations to produce an efficient solver for indefinite system of equations. We consider different preconditioner approximations of the building blocks of the preconditioner and compare their effects in accelerating the MINRES iterative scheme. We give a detailed overview of the algorithmic aspects and the theoretical convergence analysis of our solver. We study the MINRES method with the following preconditioner approximations: diagonal, multigrid v-cycle, preconditioned conjugate gradient and Chebyshev semi iteration methods. A comparative analysis of the preconditioner approximations show that the multigrid method is a suitable accelerator for the MINRES method. The application of the preconditioner becomes mandatory as evidenced by poor performance of the MINRES as compared to PMINRES. We study the problem in a two dimensional setting using the Hood-Taylor $Q_2 - Q_1$ stable pair of finite elements. The incompressible flow iterative solution software(IFISS) matlab toolbox is used to assemble the matrices. We present the numerical results to illustrate the efficiency and robustness of the MINRES scheme with the multigrid preconditioner.

Keywords: Stokes equations; Mixed finite element method; Block preconditioner; Preconditioned MINRES method(PMINRES)

NOMENCLATURE

$\hat{A}$	preconditioner for matrix A
$(A = (a_{ij})i, j, ...)$	matrices
h, l	mesh size and mesh level
$\hat{S}$	preconditioners matrix S
$\widehat{\mathcal{M}_A}$	preconditioner approximation for $\hat{A}$
$\widehat{\mathcal{M}_S}$	preconditioner approximation for $\hat{S}$
$(\mathbf{u}, ..)$	vector valued functions
$\mathbf{u} \cdot \mathbf{v}$	scalar product
$\mathbf{V}_h, W_h$	finite element spaces
$(p, q, ...)$	s scalar values
$\mathcal{R}^2$	space of dimension 2
$\Omega \in \mathcal{R}^2$	solution domain
$\partial\Omega$	boundary of Ω
$\nabla\mathbf{u} : \nabla\mathbf{v}$	componentwise scalar product
∇p	gradient of p
$\Delta\mathbf{u}$	Laplacian of $\mathbf{u}$
$div\ \mathbf{u}$	divergence of u
α, α_1	continuity constants
α_0	coercivity constant
β	continuous inf-sup constant
β_1	discrete inf-sup constant
ϕ, ψ	finite element bases functions

η	global error estimator	$H^1(\Omega), H_0^1(\Omega)$	Sobolev spaces
$\parallel \nabla\cdot\mathbf{u} \parallel_\Omega$	velocity divergence error	$Q_2 - Q_1$	biquadratic-bilinear finite eigenvalue elements
$L^2(\Omega)$	Lebesgue space of measurable functions f	$\lambda, [\omega, \bar{\omega}], [\delta, \bar{\delta}]$	eigenvalues and bounds

1. INTRODUCTION

In this paper we analyze the performance of the MINRES and the preconditioned MINRES method on the numerical solution of the steady state Stokes equations. We study Stokes equations which govern the flow of steady, viscous, incompressible, isothermal, Newtonian fluids. They arise by simplifying the incompressible Navier-Stokes equations through the omission of the convective derivative and setting the time derivative to zero. For more details on the derivation, we refer to (Bramble and Pasciak 1999; Brenner and Scott 2008; Brezinski 2005; Cebeci *et al.* 2005; Elman *et al.* 2005; Ferziger and Peric 2002; Shaughnessy *et al.* 2005; Tu *et al.* 2008; Zulehner 2002). This results in the following system of linear system partial differential equation:

$$-\triangle\mathbf{u} + \nabla p = \mathbf{f} \qquad in\ \Omega \qquad (1)$$

$$div\,\mathbf{u} = \mathbf{0} \qquad in\ \Omega \qquad (2)$$

$$\mathbf{u} = 0 \qquad on\ \partial\Omega \qquad (3)$$

where Ω is assumed to be a bounded open set in $\mathcal{R}^2$, with a sufficiently smooth and Lipschitz continuous boundary $\partial\Omega$, $\mathbf{f} : \Omega \to \mathcal{R}$ is a density of body forces acting on the fluid(e.g gravitational force) and the kinetic viscocity of the fluid has been set to 1. The problem is to find the vector function $\mathbf{u} : \overline{\Omega} \to \mathcal{R}^2$ which denotes the velocity of the fluid and $p : \Omega \to \mathcal{R}$ pressure.

The solution procedure begins with the mixed finite element discretization of the domain of the Stokes equations Eqs.(1)-(3) which results in a coupled linear algebraic systems of block structure. This study employs the Hood-Taylor $Q_2 - Q_1$ pair of finite elements as used in Elman (2007). The efficient iterative techniques for solving the linear algebraic system of the saddle point form Eq.(4) below are considered. The block coupled system has the form

$$\begin{bmatrix} A & B^T \\ B & O \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ g \end{bmatrix} \qquad (4)$$

with A- an $n \times n$ symmetric positive definite 2×2 block matrix, B- an $m \times n$ matrix with full rank and $m \leq n$, and that the coefficient matrix $\mathcal{M}$ is an $(n+m) \times (n+m)$ symmetric indefinite matrix, hence MINRES is applicable as a solution iterative scheme Paige and Saunders (1975). If these conditions on the matrices A and B above hold then the invertibility of the coefficient matrix $\mathcal{M}$ is guaranteed and that the Schur compliment form S is also symmetric positive definite (Elman *et al.* 2005; Elman and Golub 1994; Reusken and Gross 2011; Zulehner 2000; Zulehner 2002). Since matrix $\mathcal{M}$ is very large, sparse and indefinite well established iterative schemes such as Krylov subspace methods become very slow, stagnant or fail to converge if not conveniently preconditioned. The choice of the preconditioner is such that the preconditioned matrix satisfies properties needed for the numerical scheme. For the MINRES it means that the preconditioner should be symmetric and positive definite to preserve the symmetry of the preconditioned system. Preconditioning enhances the convergence behavior of the iterative schemes. There are a number of solution schemes for finding the solution ($\mathbf{u}$,p) of Eq. (4) and provided the preconditioners for A and S are at available. A variety of preconditioned iterative schemes exist to tackle the system. The different variants of preconditioners were developed thereafter. This study analyses the performance and the theoretical behaviour of preconditioning strategies and preconditioner approximations used with Krylov subspace iteration to obtain the solution of the system Eq. (4). For this class of systems, special iterative schemes must be designed because of their indefiniteness and poor spectral properties. A very extensive survey of these schemes is given in (Benzi *et al.* 2005; Elman *et al.* 1999; Turek 1999) among others. Some recent studies on the numerical solution of such system includes (Herzog and Sachs 2005; Peters *et al.* 2005; Rehman and Vuik 2007; Stoll and Wathen 2008; Vuik 1996; **?**; Zulehner 2000; Zulehner 2002). The candidate method of choice for symmetric indefinite system is the MINRES introduced in Paige and Saunders (1975) as a method of minimizing the residual $\parallel r_i \parallel_2 = \parallel Mx_i - b \parallel_2$ over the current Krylov subspace $span(r_0, Mr_0, M^2r_0, ..., M^{i-1}r_0)$. In this study we consider the MINRES method with a block diagonal preconditioner which is symmetric positive definite. In order to apply the MINRES method we need the preconditioner to be symmetric and positive definite and hence the block diagonal preconditioners would present the natural choice Elman *et al.* (1999). In this paper we apply the MINRES scheme with a block diagonal preconditioner

$$\widehat{\mathcal{M}} = \begin{bmatrix} \widehat{A} & O \\ O & \widehat{S} \end{bmatrix}$$

with $\widehat{A}$ and $\widehat{S}$ the preconditioner of the (1,1)-A block and the Schur compliment respectively. The PMINRES and its variants has been investigated in (Elman *et al.* 2005; Larin and Reusken 2008; Peters *et al.* 2005; Rees and Stoll 2010; Stoll and Wathen 2008; Wathen and Rees 2009). This study is influenced by the work by (Elman *et al.* 1999; Rees and Stoll 2010; Stoll and Wathen 2008) and the variant of the method considered in this study is the one implemented as a driver in matlab.

In this study the velocity is preconditioned by applying multigrid v-cycle method and for the pressure, its preconditioner is an iterative solver using the pressure mass matrix. The main point of this preconditioner is to reduce the low frequency error component on a coarser grid, which the high frequency components of error are reduced by a smoother on a fine grid. The multigrid method has been studied in (Brenner and Scott 2008; Hackbucsh 1985; Wesseling 1999). The multigrid are the most effective methods for solving large linear systems arising from the elliptic partial differential equations. The philosophy of the multigrid is based on the combination of two key principles. The first is that the high frequency components of the error are reduced by applying the basic iterative solvers like Jacobi or Gauss Seidel as smoothers. Next, the low frequency errors are reduced by coarse grid correction procedure. The smooth error components are represented as a solution of an approximate coarser system. After solving the coarser problem, the solution is interpolated back to correct the fine grid approximation for its low frequency errors. The way the multigrid components are linked that is smoothing, restriction, prolongation and the error of the coarse grid is illustrated in the algorithm below.

Algorithm 1. *Solve $A_h u_h = b_h$ where the subscript is used for fine grid and H for coarse rid.*

- *Perform smoothing by using k_1 iterations of an iterative method (Jacobi, Gauss-Seidel etc) on the problem $A_h u_h = b_h$*
- *Compute the residual $r_h = b_h - A_h u_h$*
- *Solve for the coarse grid correction $A_H e_H = b_H$*
- *Prolongate and update $u_h = u_h + Pe_H$*
- *Perform smoothing by using k_2 iterations of an iterative method (Jacobi, Gauss-Seidel etc) on the problem $A_h u_h = b_h$*

The algorithm is a two grid method but step 4 can be utilised in various ways. The classical method of solving it employs recursive calls to the two grid method. If the recursion is carried out in a loop, allowing different numbers of iterative sweeps on different coarse grids, we obtain v-cycle, w-cycle and f-cycle. In this study we use the multigrid v-cycle with Gauss-Seidel smoother as preconditioner approximation for both A and S.

The rest of the paper is organized as follows. In section 2 the discrete system of the Stokes equations discretized by mixed finite element method is discussed. In section 3 the iterative solution technique: The MINRES algorithm and the PMINRES algorithm together with the block preconditioner are outlined. The method relies on good approximations to the (1,1)-block and Schur complement. We also discuss the suitable bounds and the known theoretical convergence analysis based on the eigenvalue problem and results. In the final section 4 a numerical experimental results and comparative analysis on the effectiveness of the preconditioner approximations are presented, discussed and the conclusion given in section 5.

2. THE DISCRETE STOKES EQUATIONS

For the discretization of the Stokes equations we need to transform the system Eqs. (1)-(3) firstly to the variational form. The variational formulation of the Stokes equations requires that we define the following solution and test spaces:

$$H^1(\Omega) := \{\mathbf{u} : \Omega \longrightarrow \mathcal{R} | \ \mathbf{u}, \ \nabla\mathbf{u} \in L^2(\Omega)\}$$
$$H_0^1(\Omega) := \{\mathbf{v} : \ \mathbf{v} \in H^1(\Omega) | \ \mathbf{v} = 0 \ on \ \partial\Omega\}.$$

By multiplication of the first Eq. (1) with $\mathbf{v} \in H_0^1(\Omega)$

and the second Eq. (2) with $q \in L^2(\Omega)$, subsequently integrating over the domain Ω, applying the Gauss theorem, and incorporating the boundary condition Eq. (3), we obtain the variational form

Find $\mathbf{u} \in H_0^1(\Omega)$ and $p \in L^2(\Omega)$ such that:

$$a(\mathbf{u},\mathbf{v}) - b(\mathbf{v},p) = F(\mathbf{v}) \;\; \forall\, \mathbf{v} \in H_0^1(\Omega) \tag{5}$$

$$b(\mathbf{u},q) = 0 \qquad \forall\, q \in L^2(\Omega). \tag{6}$$

where a(.,.) and b(.,.) are continuous bilinear forms defined as

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} dx \tag{7}$$

$$b(\mathbf{u},q) = \int_\Omega (div\ \mathbf{v}) q dx \tag{8}$$

$$F(\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v} dx \tag{9}$$

where $\nabla\mathbf{u} : \nabla\mathbf{v}$ represents a componentwise scalar product that is $\nabla u_x \cdot \nabla v_x + \nabla u_y \cdot \nabla v_y$ and $a : H_0^1(\Omega) \times H_0^1(\Omega) \longrightarrow \mathcal{R}$ and $b : H_0^1(\Omega) \times L^2(\Omega) \longrightarrow \mathcal{R}$. The well-posedness follows from the coercivity of a(.,.) in the Lax-Milgram theorem (Ciarlet (1978), Gockenbach (2006)) and partly from the inf-sup condition (Braess (2007), Brenner and Scott (2008), Elakkad *et al.* (2010), Elman *et al.* (2005), Girault and Raviart (1986), Gunzburger (1989)). Below is a sketch of the analysis of the existence uniqueness and stability of the solution $(\mathbf{u},p) \in \mathbf{V} \times W$ of mixed problem Eqs. (5) and (6) with $\mathbf{V} = H_0^1$ *and* $W = L^2(\Omega)$:

i. the bilinear form a(.,.) is bounded or continuous:

$$| a(\mathbf{u},\mathbf{v}) | \leq \alpha \| \mathbf{u} \|_\mathbf{V} \cdot \| \mathbf{v} \|_\mathbf{V} \;\text{ for all } \mathbf{u},\mathbf{v} \in \mathbf{V} \textit{ and } \alpha \in \mathcal{R}$$

ii. the bilinear form a(.,.) is coercive on $\mathbf{V} := H_0^1(\Omega)$ that is there exists a positive constant α_0 : $a(\mathbf{v},\mathbf{v}) \geq \alpha_0 \| \mathbf{v} \|_\mathbf{V}^2 \;\; for\ all\ \ \mathbf{v} \in \mathbf{V}$.

iii. the bilinear a(.,.) is symmetric and non-negative

$$a(\mathbf{u},\mathbf{v}) = a(\mathbf{v},\mathbf{u}) \text{ and}$$
$$a(\mathbf{v},\mathbf{v}) \geq 0\ for\ all\ \mathbf{u},\mathbf{v} \in \mathbf{V}$$

iv. the bilinear form b(.,.) is bounded:

$$| b(\mathbf{u},q) | \leq \alpha_1 \| \mathbf{u} \|_\mathbf{V} \cdot \| q \|_W \;\text{ for all } \mathbf{u} \in \mathbf{V}, q \in W \textit{ and } \alpha_1 \in \mathcal{R}$$

v. the bilinear form b(.,.) satisfies the inf-sup condition, that is:
there exists a constant β :

$$\inf_{0 \neq q \in W} \sup_{0 \neq \mathbf{v} \in \mathbf{V}} \frac{| b(\mathbf{v},q) |}{\| \mathbf{v} \|_\mathbf{V} \cdot \| q \|_W} \geq \beta > 0$$

For instance in Girault and Raviart (1986) it is shown that in our concrete case b(.,.) fulfills the inf-sup condition, thus we can combine the assumptions above to give the following theorem

Theorem 2. *Variational problem Eqs. (5) and (6) is uniquely solvable provide properties (i)-(v) are all satisfied.*

The details of the proof can be found in (Girault and Raviart 1986; Brezinski 2005).

2.1 The Mixed Finite Element Discretization

The mixed finite element discretization of the variational formulation of the Stokes equations results in a linear algebraic system of equations. The finite element method described here is based on the references (Braess 2007; Brenner and Scott 2008; Ciarlet 1978; Donea and Huerta 2003; A. Elakkad and A. Elkhalfi and N. Guessous 2010; Gunzburger 1989). We will introduce the concept of mixed finite element methods. Details can be found in (Brenner and Scott 2008; Ciarlet 1978; Donea and Huerta 2003; A. Elakkad and A. Elkhalfi and N. Guessous 2010; Gockenbach 2006).

We assume that $\Omega \subseteq \mathcal{R}^2$. We define the following finite dimensional spaces .
Let W_h and $\mathbf{V}_h$ be subspaces of W and $\mathbf{V}$ respectively and now we can formulate a discrete version of problem Eqs. (5) and (6):
Find a couple $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ such that

$$a(\mathbf{u}_h,\mathbf{v}_h) - b(\mathbf{v}_h,p_h) = F(\mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h \tag{10}$$

$$b(\mathbf{u}_h,q_h) = 0, \qquad \forall q_h \in W_h. \tag{11}$$

The finite element discretization should satisfy the discrete inf-sup condition. The following theorem shows that again the inf-sup condition is of major importance.

Theorem 3. *Assume that a is V_h-elliptic (with h independent ellipticity constant) and that there exists a constant $\beta > 0$ (independent of h) such that the discrete inf-sup condition*

$$\inf_{0 \neq q_h \in W_h} \sup_{0 \neq \boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{| b(\boldsymbol{v}_h,q_h) |}{\| \boldsymbol{v}_h \|_{\boldsymbol{V}_h} \| q_h \|_{W_h}} \geq \beta > 0. \tag{12}$$

holds. Then the associated (discretized, steady state) Stokes problem has a unique solution $(\boldsymbol{u}_h; p_h)$, and there exists a constant β_1 such that

$$\| \boldsymbol{u}-\boldsymbol{u}_h \|_V + \| p-p_h \|_W \leq \beta_1 (\inf_{\boldsymbol{v}\in \boldsymbol{V}_h} \| \boldsymbol{u}-\boldsymbol{u}_h \|_V) + (\inf_{\boldsymbol{q}\in \boldsymbol{W}_h} \| p-p_h \|_W). \quad (13)$$

(for the proof we refer to Girault and Raviart (1986)).

If the basis of W_h is given by $\{\psi_1,...,\psi_m\}$ and of V_h be given by $\{\varphi_1,...,\varphi_n\}$ then

$$\mathbf{u}_h = \sum_{i=1}^{n_i} \mathbf{u}_i \cdot \varphi_i + \sum_{i=n_i+1}^{n_i+n_\partial} \mathbf{u}_i \cdot \varphi_i, \quad (14)$$

$$p_h = \sum_{k=1}^{m} p_k \psi_k. \quad (15)$$

where n_i is the number of inner nodes, n_∂ is the number of boundary nodes so the coefficients $\mathbf{u}_i : i = n_i+1,..,n_i+n_\partial$ interpolates the boundary data and $n = n_i + n_\partial$. The mixed finite element entails partitioning of the solution domain Ω into quadrilaterals, in our case that is $\Omega = \cup_i \tau_i$ we denote a set of quadrilateral elements by $T_h = \{\tau_1, \tau_2, \tau_3, ...\}$ and on each element τ_i and we denote the space $P_k(\tau_i)$ of degree less than or equal to k. There are a variety of finite element pairs whose effectiveness is through stabilization[14]. In this work we are going to use Hood-Taylor $Q_2 - Q_1$ pair of quadrilateral finite elements which are known to be stable.

We specify

$$\mathbf{V}_h := \{\mathbf{u}_h \in \mathbf{V} \mid \mathbf{u}_h |_{\tau_i} \in P_2(\tau_i), \forall \ elements \ \tau_i\},$$
$$\mathbf{W}_h := \{p_h \in \mathbf{W} \mid p_h |_{\tau_i} \in P_1(\tau_i), \forall \ elements \ \tau_i\},$$

An element $(\mathbf{u}_h, p_h) \in W_h \times V_h$ is uniquely determined by specifying components of vector $\mathbf{u}_h$ on the nodes and on the midpoints of the edges of the elements and the values of p_h on the nodes of the elements. The mixed finite element method results in the coupled linear algebraic system of equations which has to be solved by the appropriate solvers. The resulting coupled system is

$$\begin{bmatrix} A_h & B_h^T \\ B_h & O \end{bmatrix} \begin{bmatrix} \mathbf{u}_h \\ p_h \end{bmatrix} = \begin{bmatrix} \mathbf{f}_h \\ g_h \end{bmatrix} \quad (16)$$

with A_h is a block Laplacian matrix and B_h is the divergence matrix whose entries are given by

$$A = [a_{ij}], \qquad a_{ij} = \int_\Omega (\nabla\phi_i : \nabla\phi_j)_{i,j=1,...,n}$$

$$B = [b_{ki}], \qquad b_{ki} = -\int_\Omega (\psi_k \nabla \cdot \phi_i))_{k=1,...,m;\ i=1,...,n}$$

The entries of the right hand side vector are

$$\mathbf{f} = [\mathbf{f}_i], \qquad \mathbf{f}_i = \int_{\partial\Omega} \mathbf{f} \cdot \phi_i - \sum_{i=n+1}^{n+n_\partial} \mathbf{u}_i \int_\Omega (\nabla\phi_i : \nabla\phi_j)$$

$$g = [g_k], \qquad g_k = \sum_{i=n+1}^{n+n_\partial} \mathbf{u}_i \int_\Omega (\psi_k \nabla \cdot \phi_i))$$

The linear algebraic system can be represented as

$$\mathcal{M} x = b, \quad (17)$$

where $\mathcal{M} := \begin{bmatrix} A_h & B_h^T \\ B_h & O \end{bmatrix}$; $x := \begin{bmatrix} \mathbf{u}_h \\ p_h \end{bmatrix}$ and $b := \begin{bmatrix} \mathbf{f}_h \\ g_h \end{bmatrix}$.

The solution vectors $(\mathbf{u}_h, p_h)$ are the mixed finite element weak solution. The system Eqs. (16)-(17) is called the discrete Stokes problem.

The discretization and assembling of matrices are done by the matlab implementation of the mixed finite element method (Elman *et al.* 2005). A_h is stiffness matrix resulting from the discretization of the Laplacian. The resultant coefficient matrix is large, sparse, indefinite and the system must be solved iteratively, in this case by MINRES solvers. We are interested in the approximate solution of Eq. (16) at the finest mesh/discretization.

3. THE PRECONDITIONED MINRES ITERATIVE SCHEME

In this section we outline the algorithmic structure of the iterative scheme for solving the discretized linear algebraic system Eq. (16). The main goal is to find the approximate pair $(\mathbf{u}_h, p_h)$ of the discrete velocity and the discrete pressure variables at the level of refinement l. To calculate this, we apply our solvers to the discrete algebraic system of equations. The algorithmic details are outlined for the MINRES and PMINRES iteration. The solvers have the appeal of not requiring any parameters for making the algorithm efficient as compared to other methods like non-standard conjugate gradient method and other variants of the Uzawa methods. Elman *et al.* (1999) also recommended the preconditioned MINRES to solve the saddle point problems. For convenience, we consider Eq. (17). The MINRES method is based on the following residual minimization problem:

Given the initial guess x_0, determine $x_i \in x_0 + \mathcal{K}^i(\mathcal{M};r_0)$ such that $\| \mathcal{M}x_i - b \| = min(\| \mathcal{M}x_i - b \| | x_0 + \mathcal{K}^i(\mathcal{M};r_0))$ where $r_0 = b - \mathcal{M}x_0$ and $\mathcal{K}^i(\mathcal{M},r_0) = span\left\{r_0, \mathcal{M}r_0, \mathcal{M}^2 r_0, ..., \mathcal{M}^{i-1}r_0\right\}$ is the Krylov subspace. Below is the MINRES algorithm for computing the iterate x^i as given in (Elman *et al.* (1999)).

Algorithm 4. ***The MINRES Algorithm***

$$\boldsymbol{v}_0 = \boldsymbol{0}, \quad \boldsymbol{w}_0 = \boldsymbol{0}, \quad \boldsymbol{w}_1 = \boldsymbol{0}$$
Choose $\boldsymbol{x}_0$, *compute*
$$\boldsymbol{v}_1 = \boldsymbol{b} - \mathcal{M}\boldsymbol{x}_0, \quad set\ \gamma_1 = \| \boldsymbol{v}_1 \|$$
$$set\ \eta = \gamma_1,\ s_0 = s_1 = 0, \quad c_0 = c_1 = 1$$
for i = 1,2...until convergence do
$$\boldsymbol{v}_i = \frac{\boldsymbol{v}_i}{\gamma_i}$$
$$\delta_i = \langle \mathcal{M}\boldsymbol{v}_i, \boldsymbol{v}_i \rangle$$
$$\boldsymbol{v}_{i+1} = \mathcal{M}\boldsymbol{v}_i - \delta_i \boldsymbol{v}_i - \gamma_i \boldsymbol{v}_{i-1}$$
$$\gamma_{i+1} = \| \boldsymbol{v}_{i+1} \|$$
$$\alpha_0 = c_i\delta_i - c_{i-1}s_i\gamma_i$$
$$\alpha_1 = \sqrt{\alpha_0^2 + \gamma_{i+1}^2}$$
$$\alpha_2 = s_i\delta_i + c_{i-1}c_i\gamma_i$$
$$\alpha_3 = s_{i-1}\gamma_i$$
$$c_{i+1} = \frac{\alpha_0}{\alpha_1}, \quad s_{i+1} = \frac{\gamma_{i+1}}{\alpha_1}$$
$$\boldsymbol{w}_{i+1} = \frac{(\boldsymbol{v}_i - \alpha_3 \boldsymbol{w}_{i+1} - \alpha_2 \boldsymbol{w}_j)}{\alpha_1}$$
$$\boldsymbol{x}_i = \boldsymbol{x}_{i-1} + c_{i+1}\eta \boldsymbol{w}_{i+1}$$
$$\eta = -s_{i+1}\eta$$
Test for convergence
end for

We consider a symmetric positive definite block preconditioner $\widehat{\mathcal{M}}$ of $\mathcal{M}$ as studied in (Schorbel and Zulehner (2007), Zulehner (2002)). The block triangular preconditioner is given as

$$\widehat{\mathcal{M}} := \begin{bmatrix} \widehat{A} & O \\ O & \widehat{S} \end{bmatrix} \tag{18}$$

Such that the left preconditioned system becomes

$$\widehat{\mathcal{M}}^{-1}\mathcal{M}x = \widehat{\mathcal{M}}^{-1}b \tag{19}$$

which is

$$\widetilde{\mathcal{M}}x = \widetilde{b} \tag{20}$$

The residual minimization criteria is applied to the preconditioned system Eq. (20):

Given the initial guess $x_0 \in \mathcal{R}^{n+m\times n+m}$ and initial residual $\widetilde{r}_0 = \widetilde{b} - \widetilde{\mathcal{M}}x_0$, determine $x_i \in x_0 + \mathcal{K}^i(\widetilde{\mathcal{M}};\widetilde{r}_0)$ such that: $\| \widetilde{\mathcal{M}}x_i - \widetilde{b} \| = min(\| \widehat{\mathcal{M}}x_i - \widetilde{b} \| \quad | x_0 + \mathcal{K}^i(\widetilde{\mathcal{M}};\widetilde{r}_0))$ where $\widetilde{r}_0 = \widetilde{b} - \widetilde{\mathcal{M}}x_0$ and $\mathcal{K}^i(\widetilde{\mathcal{M}},\widetilde{r}_0) = span\left\{\widetilde{r}_0, \widetilde{\mathcal{M}}\widetilde{r}_0, \widetilde{\mathcal{M}}^2\widetilde{r}_0, ..., \widetilde{\mathcal{M}}^{i-1}\widetilde{r}_0\right\}$ is the Krylov subspace and $\langle\cdot,\cdot\rangle_{\widehat{M}} = \langle \widehat{M}\cdot,\cdot\rangle$.

Then the preconditioned residual $\widehat{\mathcal{M}}^{-1}(b - \mathcal{M}x)$ is minimized in $\| \cdot \|_{\widehat{M}}$ over transformed Krylov subspace. The implementation of the PMINRES requires one evaluation of $\widehat{\mathcal{M}}^{-1}z$ for a given z and one multiplication by A per iteration. The evaluation of the preconditioner is achieved by solving a linear system $z = \widehat{\mathcal{M}}y$

The PMINRES method is outlined in the algorithm below.

Algorithm 5. ***The PMINRES Algorithm***

$$\boldsymbol{v}_0 = \boldsymbol{0}, \quad \boldsymbol{w}_0 = \boldsymbol{0}, \quad \boldsymbol{w}_1 = \boldsymbol{0}$$
Choose $\boldsymbol{x}_0$, *compute*
$$\boldsymbol{v}_1 = \boldsymbol{b} - \mathcal{M}\boldsymbol{x}_0, \quad set\ \gamma_1 = \| \boldsymbol{v}_1 \|$$
$$Solve\ \widehat{M}\boldsymbol{z}_1 = \boldsymbol{v}_1 \quad set\ \gamma_1 = \sqrt{\langle \boldsymbol{z}_1, \boldsymbol{v}_1 \rangle}$$
$$set\ \eta = \gamma_1,\ s_0 = s_1 = 0, \quad c_0 = c_1 = 1$$
for i = 1,2...until convergence do
$$\boldsymbol{z}_i = \frac{\boldsymbol{z}_i}{\gamma_i}$$
$$\delta_i = \langle \mathcal{M}\boldsymbol{z}_i, \boldsymbol{z}_i \rangle$$
$$\boldsymbol{v}_{i+1} = \mathcal{M}\boldsymbol{v}_i - \left(\frac{\delta_i}{\gamma_i}\right)\boldsymbol{v}_i - \left(\frac{\gamma_i}{\gamma_{i-1}}\right)\boldsymbol{v}_{i-1}$$
$$Solve\ \widehat{M}\boldsymbol{z}_{i+1} = \boldsymbol{v}_{i+1}$$
$$\gamma_1 = \sqrt{\langle \boldsymbol{z}_{i+1}, \boldsymbol{v}_{i+1} \rangle}$$
$$\alpha_0 = c_i\delta_i - c_{i-1}s_i\gamma_i$$
$$\alpha_1 = \sqrt{\alpha_0^2 + \gamma_{i+1}^2}$$
$$\alpha_2 = s_i\delta_i + c_{i-1}c_i\gamma_i$$
$$\alpha_3 = s_{i-1}\gamma_i$$
$$c_{i+1} = \frac{\alpha_0}{\alpha_1}, \quad s_{i+1} = \frac{\gamma_{i+1}}{\alpha_1}$$
$$\boldsymbol{w}_{i+1} = \frac{(\boldsymbol{z}_i - \alpha_3 \boldsymbol{w}_{i+1} - \alpha_2 \boldsymbol{w}_j)}{\alpha_1}$$
$$\boldsymbol{x}_i = \boldsymbol{x}_{i-1} + c_{i+1}\eta \boldsymbol{w}_{i+1}$$
$$\eta = -s_{i+1}\eta$$
Test for convergence
end for

The main convergence results for PMINRES method are due to (Peters *et al.* 2005; Rees and Stoll 2010; Reusken and Gross 2011). The convergence analysis is based on the eigenvalue analysis of the preconditioned matrix system. The spectral analysis of the preconditioner influences the convergence properties of the iterative scheme connected with the approximations of $\widehat{A}$ *and* $\widehat{S}$. We require that the eigenvalues of the preconditioned system are well clustered and distributed provided that the eigenvalues of $\widehat{A}^{-1}A$ *and* $\widehat{S}^{-1}S$ are so. The convergence of the iterative scheme is driven mostly by the ratio between the largest and smallest eigenvalue of the preconditioned system. The theorem below gives convergent analysis of the PMINRES method

Theorem 6. *Let $\mathcal{M} \in \mathcal{R}^{(n+m)\times(n+m)}$ be symmetric and $\widehat{M} \in \mathcal{R}^{(n+m)\times(n+m)}$ be symmetric and positive definite. For $x_i,\ i > 0$ computed in the preconditioned MINRES algorithm we define $\widehat{r}_i = \widehat{M}^{-1}(b - \mathcal{M}x_i)$. Then the following holds*

$$\| \widehat{r}_i \|_{\widehat{M}} = min_{p_i \in \mathcal{P}_i; p_0=1} \| p_i(\widehat{M}^{-1}\mathcal{M})\widehat{r}_0 \|_{\widehat{M}} \leq min_{p_i \in \mathcal{P}_i; p_0=1} max_{\lambda \in \sigma(\widehat{M}^{-1}\mathcal{M})} |p_i(\lambda)| \| \widehat{r}_0 \|_{\widehat{M}} \quad (21)$$

The maximum is over the eigenvalues λ of $\widehat{\mathcal{M}}^{-1}\mathcal{M}$.

For the proof we refer to (Reusken and Gross (2011)). The theorem gives that the rate of convergence of the preconditioned MINRES method depends on $\sigma(\widehat{M}^{-1}\mathcal{M})$. We need to derive these bounds. The special case is when we have $\widehat{A} = A$ *and* $\widehat{S} = S$ and we have that $\sigma(\widehat{M}^{-1}\mathcal{M}) \in \{\frac{1}{2}(1-\sqrt{5}), 1, \frac{1}{2}(1+\sqrt{5})\}$. This yields an exact solution in at most 3 iterations (Elman *et al.* 1999; Peters *et al.* 2005; Reusken and Gross 2011). In our case this is not practically feasible since the Schur compliment $\widehat{S} = B\widehat{A}^{-1}B^T$ is not computed explicitly since $\widehat{A}^{-1}$ is not sparse anymore and is very costly to use since the matrix vector computation $\widehat{S}z = B(\widehat{A}^{-1}(B^Tz))$ that involves solving a linear system like $\widehat{A}y = B^Tz$. Since the exact preconditioning is not feasible, we use the preconditioner approximations $\widehat{\mathcal{M}_A}$ *of* $\widehat{A}$ *and* $\widehat{\mathcal{M}_S}$ *of* $\widehat{S}$. The quality of these approximations is measured by using the spectral equivalences or inequalities. In order to quantify the quality of the both approximations we use the spectral analysis under the following conditions. For the preconditioner approximations $\widehat{\mathcal{M}_A}$ *and* $\widehat{\mathcal{M}_S}$ and let $\omega,\ \overline{\omega},\ \delta,\ \overline{\delta} > 0$ such that

$$\omega \leq \lambda = \frac{x^TAx}{x^T\widehat{\mathcal{M}_A}x} \leq \overline{\omega} \quad (22)$$

$$\delta = \frac{x^TBA^{-1}B^Tx}{x^T\widehat{\mathcal{M}_S}x} \leq \overline{\delta} \quad (23)$$

Using the results in (Elman *et al.* 1999; Peters *et al.* 2005; Rees and Stoll 2010) we obtain the result for the eigenvalue bounds for the preconditioned matrix. The conditions Eqs. (22) and (23) implies that $[\omega, \overline{\omega}]$ contain all the eigenvalues of the preconditioned matrix $\widehat{A}^{-1}A$ and $\overline{\delta}$ is the smallest eigenvalue of $\widehat{S}^{-1}S$, with δ the inf-sup constant. As a consequence the following theorem for eigenvalue bounds points to the convergence of PMINRES method.

Theorem 7. *Let $\omega,\ \overline{\omega},\ \delta,\ \overline{\delta} > 0$ and assume that Eqs. (22) and (23). Let $(\lambda, \begin{bmatrix} u \\ p \end{bmatrix})$ be an eigenpair $\widetilde{\mathcal{M}}$. Then following eigenvalue problem*

$$\mathcal{M}\begin{bmatrix} u \\ p \end{bmatrix} = \lambda \begin{pmatrix} \widehat{\mathcal{M}_A} & O \\ O & \widehat{\mathcal{M}_S} \end{pmatrix} \begin{bmatrix} u \\ p \end{bmatrix} \quad (24)$$

and $A, S, \widehat{M}_A$ and $\widehat{M}_S$ is positive definite. Then λ is real and positive that satisfies

$$\frac{\omega - \sqrt{\omega^2 - 4(\overline{\omega}\overline{\delta})}}{2} \leq \lambda \leq \frac{\overline{\omega} - \sqrt{\overline{\delta}^2 + 4\delta\omega}}{2}$$
$$\omega \leq \lambda \leq \overline{\omega}$$
$$\frac{\omega + \sqrt{\omega^2 + 4\omega\delta}}{2} \leq \lambda \leq \frac{\overline{\omega} + \sqrt{\overline{\omega}^2 + 4(\overline{\delta}\overline{\omega})}}{2}$$

with $\omega,\ \overline{\omega},\ \delta,\ \overline{\delta}$ measures the quality and effectiveness of the preconditioners(approximations) $\widehat{M}_A$ of A and $\widehat{M}_S$ to S.

For the proof we refer to (Elman *et al.* 1999; Elman *et al.* 2005; Rees and Stoll 2010).

Remark 8. *The results of the theorems (6) and (7) indicate that as long as we choose the approximations $\widehat{\mathcal{M}_A}$ and $\widehat{\mathcal{M}_S}$ close enough to A and S then we can expect a good clustering of eigenvalues of the eigenvalue problem that is the spectral constants are $\omega,\ \overline{\omega},\ \delta,\ \overline{\delta}$ close 1 then one can expect a fast convergence of the PMINRES method.*

4. NUMERICAL RESULTS

In this section we present the numerical solution of classical Stokes problem Eqs. (1)-(3) using the solvers presented above. The solvers are denoted by MINRES (algorithm 4) and PMINRES (algorithm 5). We present the results of this method as outlined above to run the traditional test problem, the driven cavity flow problem (Bramble *et al.* 1997; Bramble and Pasciak 1999; Elman *et al.* 1999; Larin and Reusken 2008). It is a model of the flow in a square cavity (the domain is Ω) with the top lid moving from left to right in our case the regularized cavity model $\{y = 1 :\ -1 \leq x \leq 1 \mid u_x = 1 - x^4\}$ Elman *et al.* (1999). The Dirichlet no-slip boundary condition is applied on the side and bottom boundaries. The mixed finite element method was used to discretize the cavity domain $\Omega = (-1,1)^2$.

We pay particular attention to the computational performance of the MINRES and PMINRES methods at different grid levels. We compare the effectiveness of different approximations for the preconditioners $\widehat{\mathcal{M}_A}$ *and* $\widehat{\mathcal{M}_S}$. The following are considered, no preconditioning and different combinations of the preconditioners $\widehat{A}$ *and* $\widehat{S}$ are considered as outlined in section 3:

i. MINRES with no preconditioning

ii. PMINRES: diagonal preconditioning (diag(A), diag(Q)), Q is the pressure mass matrix.

iii. PMINRES: multigrid v-cycle with Gauss Seidel smoothing for $\widehat{\mathcal{M}_A}$ and multigrid v-cycle for the $\widehat{\mathcal{M}_S}$ using the pressure mass matrix (A_{mg}, S_{mg})

iv. PMINRES: multigrid v-cycle with Gauss Seidel smoothing for $\widehat{\mathcal{M}_A}$ and diagonally preconditioned standard conjugate gradient for the $\widehat{\mathcal{M}_S}$ using the pressure mass matrix (A_{mg}, S_{pcg})

v. PMINRES: multigrid v-cycle with Gauss Seidel smoothing for $\widehat{\mathcal{M}_A}$ and Chebyshev semi-iteration for the $\widehat{\mathcal{M}_S}$ using the pressure mass matrix **?** (A_{mg}, S_{cheb}).

vi. PMINRES: multigrid v-cycle with Gauss Seidel smoothing for $\widehat{\mathcal{M}_A}$ and diagonal preconditioning using the pressure mass matrix $(A_{mg}, diag(Q))$.

The comparison is made on the performance of the MINRES and PMINRES schemes with different combinations of preconditioners (i)-(vi) in terms of iterative counts and CPU time. The numerical treatment is given to the discrete Stokes problem which resulted from the mixed finite Hood-Taylor Q2-Q1 stable elements consisting of biquadratic elements for the velocities and bilinear elements for the pressure, on a uniform grid. Implementation was performed on a windows 7 platform with 2.13 GHz speed intel dual core processor by using matlab 7.14 programming language. For the discretization we start with a uniform square grid with $h_0 = \frac{1}{2}$ and we apply regular refinements to this starting discretization.

The Fig. 2 show the snapshot of the error indicators at level 6 from the MINRES and PMINRES.

In this work we use the structured mesh and regular refinements. The meshes are generated by the matlab IFISS toolbox Elman (2007) in a hierarchy of grids which are produced by successive refinements. We need to choose the coarse mesh (the starting mesh), finest mesh which corresponds to the maximum level of refinement on which the final approximate solution is considered. The assembled matrices are stored for each refinement level for the system Eq. (16). The Table (1) below shows an example of the refinement levels (in the examples below we use the coarsest (starting) level to have 9 nodes for velocity and 4 nodes for pressure variables (level 0)) but we start the computation at level 2.

The Table (1) shows the refinement levels and the number of grid points (nodes) for each level.

In all cases the iterations are repeated until the tolerance of 10^{-6} is satisfied. The schemes converge if the stopping criteria is satisfied. The Table (2) shows the numerical results of the MINRES and PMINRES (diagonal preconditioner)

The results in Table (2) show the MINRES and diagonally preconditioned MINRES are not efficient and robust in solving the Stokes equations. Hence the need for a preconditioner approximations to act as accelerators of the solution process. In applying the preconditioner, we approximate the preconditioner $\widehat{A}$ of the Laplacian stiffness and sparse matrix A by a geometric multigrid v-cycle method (A_{mg}). The multigrid is a well known fast solver for such systems. Before we apply the approximation combination we need to check the effects of the number of smoothing iterations on the performance of the multigrid preconditioner.

The Table (3) shows the effects of different multigrid v-cycle (1, 2, 3, 4) pre-and post-smoothing steps in the performance of the PMINRES method.

The results in Table (3) indicate that there is no significant difference in the performance of the PMINRES with multigrid preconditioner approximation of the preconditioner of A and Schur compliment S using pressure mass matrix Q. In the next table we present results of the PMINRES with multigrid v-cycle approximation of the Laplacian combined with different approximations of the Schur compliment using the mass pressure matrix Q.

The results in Table (4) show that the multigrid v-cycle approximation for the preconditioner $\widehat{A}$ combined with the Chebyshev semi iteration/ multigrid v-cycle for the preconditioner $\widehat{S}$ have shown better results in terms of iterative counts and faster in terms of computational time. The

Table 1 Refinement levels and number of nodes(n_l= number of velocity unknowns($\times$ 2) and m_l number of pressure unknowns)

refinement level(l)	1	2	3	4	5	6	7	8
mesh size(h_l)	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
velocity nodes(n_l)	9	25	81	289	1089	4425	16641	66049
pressure nodes(m_l)	4	9	25	81	289	1089	4425	16641

Table 2 CPU time and number of iterations for PMINRES for different preconditioner approximations at different levels of refinement, tolerance = 10^{-6}

Level	Minres	Pminres($diag(A), diag(Q)$)
	iter(cpu time (sec))	iter(cpu time (sec))
$\frac{1}{4}$	12(1.349)	12(4.4e-002)
$\frac{1}{8}$	59(1.6e-002)	33(1.9e-002)
$\frac{1}{16}$	231(1.57e-001)	86(1.00e-001)
$\frac{1}{32}$	529(8.11e-001)	182(5.78e-001)
$\frac{1}{64}$	1012(5.68)	386(4.621)
$\frac{1}{128}$	1845(4.11754e+001)	1008(5.0165e+001)

Table 3 CPU time and number of iterations for PMINRES(A_{mg}, S_{mg}) at different levels of refinement, tolerance = 10^{-6}

Levels	vmg(1,1)iter)	vmg(2,2)	vmg(3,3)	vmg(4,4)
	iter(cpu time (sec))	iter(cpu time (sec))	iter(cpu time (sec))	iter(cpu time (sec))
$\frac{1}{4}$	6(9e-003)	6(8e-003)	6(8e-003)	6(8e-003)
$\frac{1}{8}$	28(6.2e-002)	25(6.3e-002)	24(7e-002)	24(2.08e-002)
$\frac{1}{16}$	32(1.98e-001)	28(2.04e-001)	27(2.3e-001)	27(2.62e-001)
$\frac{1}{32}$	33(5,43e-001)	28(6.11e-001)	27(7.009e-001)	27(8.5e-001)
$\frac{1}{64}$	35(1.608)	30(4.621)	27(2.105)	27(2.496)
$\frac{1}{128}$	33(5.153)	28(5.017)	26(7.009)	25(8.315)

Table 4 CPU time and number of iterations for PMINRES for different preconditioner approximations at different levels of refinement, tolerance = 10^{-6}

Levels	pminres(A_{mg}, S_{mg})	pminres(A_{mg}, S_{pcg})	pminres(A_{mg}, S_{chebv})	pminres($A_{mg}, diag(Q)$)
	iter(cpu time (sec))	iter(cpu time (sec))	iter(cpu time (sec))	iter(cpu time (sec))
$\frac{1}{4}$	6(8e-003)	6(9.5e-002)	6(3.2e-002)	6(9e-003)
$\frac{1}{8}$	24(2,08e-001)	24(8.3e-002)	22(6.8-002)	21(5.6e-002)
$\frac{1}{16}$	27(2.62e-001)	29(2.81e-001)	25(2.41e-001)	40(3.24e-001)
$\frac{1}{32}$	27(8.5e-001)	29(2.9e-001)	27(8.08e-001)	44(3.4e-001)
$\frac{1}{64}$	27(2.496)	29(2.81)	27(8.08e-001)	44(1.136)
$\frac{1}{128}$	25(8.315)	29(9.252)	25(8.318)	44(1.29e+001)

Table 5 Changes in the $\| \nabla \cdot \mathbf{u} \|_\Omega$ estimated velocity divergence error. η the global error estimator from one level to the other

level	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
$\| \nabla \cdot \mathbf{u} \|_\Omega$	2.230e-001	7.6e-002	2.016e-002	5.1e-003	1.28e-003	3.2e-004
η	2.6543	1.065e+000	2.773e-001	6.87e-002	1.71e-002	4.24e-003

iterative counts and the time are more attractive as compared to the other preconditioner approximations combinations like preconditioned conjugate gradient and diagonal preconditioner of the pressure matrix. The multigrid method proves to be a suitable approximation for the matrix A from the Laplacian. This agrees with the results in Peters *et al.* (2005) and the Chebyshev iteration becomes a better approximation for the Schur compliment preconditioner using the pressure mass matrix. The results show that the PMINRES is robust because the iterative counts do not change significantly as the matrix size increases.

The Table 5 shows the changes in the estimated a posteriori errors for regularized driven cavity flow using $Q_2 - Q_1$ approximation for the flow: using the strategy built in IFISS ((Elman 2007),(Elman *et al.* 2005)) that for every element error, the local error estimation is given as by the combination of the energy norm of the velocity error and the L_2 norm of the divergence error that is

$$\eta_T^2 := \| \nabla \mathbf{e}_T \|_T^2 + \| R_T \|_T^2$$

Where $\mathbf{e}_T$ is the velocity error estimate and $R_T = \| \nabla \cdot \mathbf{u} \|_T$ and $\eta := (\sum_{T \in T_h} \eta_T^2)^{\frac{1}{2}}$ the global error estimator. From the Table 5 we note that the velocity divergence is clearly converging at a faster rate to $O(h^4)$, which means the estimated global error η is increasingly dominated by the velocity error component as $h \to 0$

The changes in the solution errors are highlighted in the table 5 below for the levels with mesh sizes $\frac{1}{4}$ to $\frac{1}{64}$.

The Fig. 1 below shows the sample snapshot of the grid output of residual reduction at the level with mesh size $\frac{1}{32}$ for MINRES and PMINRES. The Fig. 1 below the PMINRES residual error reduction is faster and is done in fewer iterations as compared to the MINRES. This is a reflection that the preconditioners were very effective in making the solver perform very well.

The most interesting observation on the figures and table 5 is that from the two iterative schemes we get the same solution and a posteriori error estimates. The above figure on the residual reduction clearly shows that the PMINRES is faster than the MINRES. This shows that the accelerator is effective in improving the performance of the MINRES method. Hence preconditioning is an effective way of improving the performance of the iterative schemes. The use of the preconditioner in mandatory in making the iterative schemes effective and robust

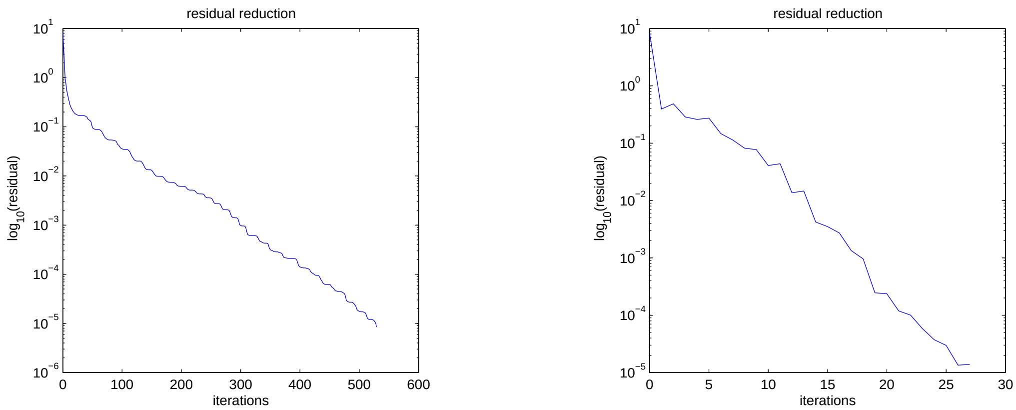

Fig. 1. Residual reduction for MINRES(left) and PMINRES(right) of the Stokes equation at level 5.

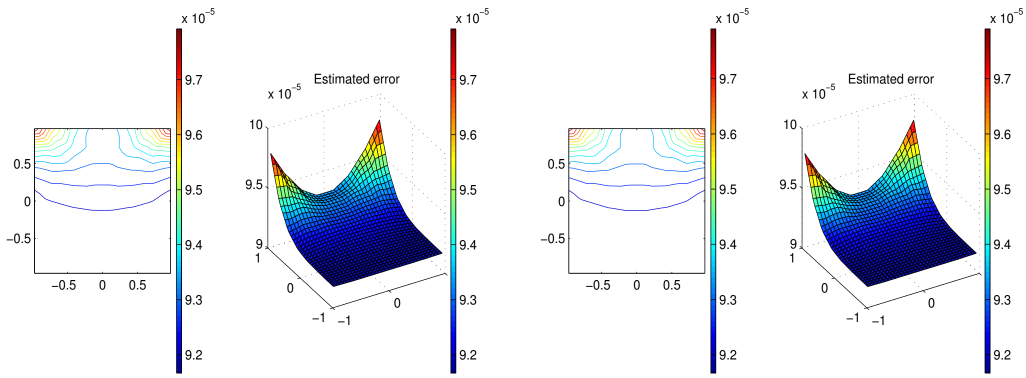

Fig. 2. Estimated error η in the computed solution at level 5 for the MINRES(left) and PMINRES (right).

5. Conclusion

The objective of the work consisted of the developing efficient and robust iterative solvers for the two dimensional steady state Stokes equations discretized by mixed finite element method $Q_2 - Q_1$ stable pair of rectangular elements. To this end, a MINRES method and its preconditioned counterpart denoted by PMINRES were used as solution schemes. In this paper we presented results for the MINRES and PMINRES that were considered through traditional benchmark lid-driven cavity domain. For PMINRES we applied a combination of the preconditioners with the main diagonal approximated by different iterative scheme. The preconditioner diagonal element $\widehat{A}$ was approximated by the multigrid v-cycle for all the tests since it is the best known solver for the Laplacian and different approximations were used for the Schur compliment. A comparative study was also made on the performance of the MINRES and PMINRES iterative schemes in terms of iterative counts and cpu time. We advocate the use of the multigrid preconditioner approximation for the Laplacian matrix A-block with the combination of the multigrid/Chebyshev semi iteration approximation for the Schur compliment to accelerate the performance of the MINRES iterative scheme. This entails that the multigrid solver is effective in accelerating the performance of the MINRES iterative scheme as justified by the numerical results. Hence it represents a robust and efficient solver of the Stokes problem. In most applications the MINRES (without preconditioning) is not an alternative since it is expensive and requires direct inversion of a huge sparse matrix.

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