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Effects of Slip Condition, Variable Viscosity and Inclined Magnetic Field on the Peristaltic Motion of a Non-Newtonian Fluid in an Inclined Asymmetric Channel

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ABSTRACT

The peristaltic motion of a third order fluid due to asymmetric waves propagating on the sidewalls of a inclined asymmetric channel is discussed. The key features of the problem includes long-wavelength and low-Reynolds number assumptions. A mathematical analysis has been carried out to investigate the effect of slip condition, variable viscosity and magnetohydrodynamics (MHD). Followed by the nondimensionalization of the nonlinear governing equations along with the non-linear boundary conditions, a perturbation analysis is made. For the validity of the approximate solution, a numerical solution is obtained using the iterative collocation technique.

Keywords: Variable viscosity; Slip conditions; Peristaltic flow; Third order fluid; Inclined asymmetric channel.

1. INTRODUCTION

During the past three decades, most of the contributions on peristaltic motion dealt with the blood and other physiological fluids as the Newtonian fluids. In the recent years, a growing interest towards the peristaltic flow of non-Newtonian fluid has been devoted by the biological and engineering practice. This approach is adequate when peristaltic mechanism involved in lymphatic vessels, small blood vessels, and transport of spermatozoa in the cervical canal is considered. It is of no surprise that majority of the physiological fluids behave like non-Newtonian fluid. (Keimanesh *et al.* 2011) studied the third grade non-newtonian fluid flow between two parallel plates. The peristaltic mechanism has been studied by several researcher experimentally, numerically and analytically (Latham 1966; Shapira *et al.* 1969; Hayat *et al.* 2008; Nadeem & Akbar 2010; Mekheimer & Elmaboud 2008; Noureen 2013; Haroun 2007; Sud *et al.* 1977; Srivastava & Agrawal 1980). Recent investigations [11,15] and many more have contributed to the study of peristaltic action.

Shapiro and Jaffrin (Shapira *et al.* 1969) discussed peristaltic pumping with long wavelength at low Reynolds number. The long wave length approximation is an assumption that the width of the channel is small compared to wavelength of peristaltic waves.

The biomagnetic fluid dynamics is the branch of fluid dynamics, which studies the peristalsis of fluids. The most characteristic biomagnetic fluid is blood. Due to the complex interaction of the intercellular protein, the hemoglobin and the red cell membrane, the blood behaves as a magnetohydrodynamic (MHD) fluid. The study of the MHD property of blood has potential for therapeutic use in the diseases of blood vessels and heart, and help in controlling blood pressure. (Rashidi *et al.* 2014) recently investigated the entropy generation in MHD and slip flow over a rotating porous disk with variable properties. The influence of moving magnetic field on blood flow was studied by Stud et'al. (Sud *et al.* 1977) and they observed that the influence of suitable moving magnetic field increases the speed of blood. Srivastava and Agrawal (Srivastava & Agrawal 1980) considered the blood as an electrically conducting fluid and constitutes a suspension of red cell in plasma. Mekheimer (Hayat *et al.* 2007) studied peristaltic flow of a magnetic field in non-uniform channels. Hayat et al. (Akram & Nadeem 2013) considered the effect of magnetic field on the peristaltic flow of a third order fluid in a symmetric channel. Akram and Nadeem (Busuioc *et al.*) presented the influence of induced magnetic field and heat transfer on the peristaltic transport of a Jeffrey's fluid in an asymmetric channel. (Rashidi *et al.* 2012) studied the entropy generation in MHD and slip flow over a rotating porous disk. (Khan *et al.* 2014) recently studied effect of heat transfer on peristaltic motion of Oldroyd fluid in the presence of inclined magnetic field.

In real system there is always certain amount of slip and no slip conditions are no longer valid. Many investigations (Nadeem & Akram 2010) have been carried out to see the effect of slip on the peristaltic motion of fluids. Moreover, the peristaltic flow propagates in symmetric as well as asymmetric channels (where the peristaltic waves propagate with different amplitude and phase on the channel walls).

All the above mentioned studies indicate that much attention in the past has been given to discuss the peristaltic flow of fluids with constant viscosity. Few attempts have been made to discuss the influence of variable viscosity on the peristaltic flow (see (Khan *et al.* 2012) and the references therein), however the present paper discusses the effects of variable viscosity and slip conditions on the peristaltic flow of a third order fluid in an inclined asymmetric channel. Analysis has been carried out in the presence of inclined magnetic field. The paper is organised as follows, in section 2, we have presented a detailed mathematical formulation of the problem. In section 3, the series solution, using the perturbation analysis is presented. In section 4, the numerical approach is outlined. In section 5, the results are discussed with the aid of graphical representation. In section 6, the conclusions are made, the benefit of the current research is discussed, and the future work is proposed.

2. MATHEMATICAL FORMULATION

Let us investigate the flow of an incompressible third-order fluid with variable viscosity in an inclined asymmetric channel of width $d_1 + d_2$. The fluid under consideration is conducting in the presence of an applied magnetic field B_0. Both the magnetic field and channel are inclined at an angle θ and γ respectively. A schematic diagram is presented in Fig. 1 to visualize the problem. The sinusoidal waves travelling down its walls are defined as

$$H_1\left(\overline{X},\bar{t}\right) = d_1 + a_1\cos\left[\frac{2\pi}{\lambda}(\overline{X}-c\bar{t})\right]$$

$$H_2\left(\overline{X},\bar{t}\right) = -d_2 - b_1\cos\left[\frac{2\pi}{\lambda}(\overline{X}-c\bar{t})+\phi\right] \quad (1)$$

In the above equations a_1, b_1 are the wave amplitudes, λ is the wavelength and ϕ $(0 \leq \phi \leq \pi)$ is the phase difference. $\phi = 0$ corresponds to symmetric channel with waves out of phase and $\phi = \pi$, the waves are in phase.

The equations of motion are given by

$$\frac{\partial\overline{U}}{\partial\overline{X}} + \frac{\partial\overline{U}}{\partial\overline{Y}} = 0, \quad (2)$$

Let $D_1 = \left(\frac{\partial}{\partial\bar{t}} + \overline{U}\frac{\partial}{\partial\overline{X}} + \overline{V}\frac{\partial}{\partial\overline{Y}}\right)$

$$\begin{aligned}\rho D_1\overline{U} \;=\;& -\frac{\partial\overline{P}}{\partial\overline{X}} + \frac{\partial\overline{S}_{\overline{XX}}}{\partial\overline{X}} + \frac{\partial\overline{S}_{\overline{XY}}}{\partial\overline{Y}} + \rho g\sin\gamma \\ & -\sigma B_0^2\cos\theta(\overline{U}\cos\theta - \overline{V}\sin\theta), \quad (3)\end{aligned}$$

$$\begin{aligned}\rho D_1\overline{V} \;=\;& -\frac{\partial\overline{P}}{\partial\overline{Y}} + \frac{\partial\overline{S}_{\overline{XY}}}{\partial\overline{X}} + \frac{\partial\overline{S}_{\overline{YY}}}{\partial\overline{Y}} - \rho g\cos\gamma \\ & +\sigma B_0^2\sin\theta(\overline{U}\cos\theta - \overline{V}\sin\theta), \quad (4)\end{aligned}$$

in which $\overline{U}$, $\overline{V}$ are the velocity in the fixed frame, ρ is constant density, $\overline{P}$ is the pressure, σ is the electrical conductivity and g is the acceleration due to gravity. Expression for extra stress tensor $\overline{S}$ in a third order fluid is defined by the following relation

$$\begin{aligned}\overline{S} \;=\;& \left(\mu + \beta_3 tr\overline{A}_1^2\right)\overline{A}_1 + \alpha_1\overline{A}_2 + \alpha_2\overline{A}_1^2 + \beta_1\overline{A}_3 \\ & +\beta_3\left(\overline{A}_1\overline{A}_2 + \overline{A}_2\overline{A}_1\right), \quad (5)\end{aligned}$$

where μ is the coefficient of shear viscosity, α_i $(i = 1,2)$ and β_i $(i = 1,2,3)$ are the material constants. (Busuioc *et al.*) proved a mathematical theory of existence and uniqueness of solutions for the constitutive law 5 with some restrictions on the material coefficients. In this discussion, we have considered some limitations on these coefficients, in a manner similar

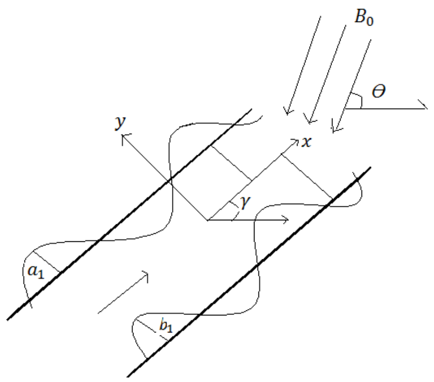

Fig. 1. Schematic diagram of the inclined flow.

to (Busuioc *et al.* ; Ali *et al.* 2009). The first three Rivlin Ericksen tensors can be written as

$$\overline{A}_1 = \overline{L}+\overline{L}^t,$$
$$\overline{A}_{n+1} = \frac{d\overline{A}_n}{d\overline{t}}+\overline{A}_n\overline{L}+\overline{L}^t\overline{A}_n, \quad n=1,2.$$

where $\overline{L}=\nabla\overline{V}$ and $\overline{t}$ represent transpose.

The coordinates and velocities in the laboratory $\left(\overline{X},\overline{Y}\right)$ and wave $(\overline{x},\overline{y})$ frames are related by the following equations

$$\overline{x}=\overline{X}-c\overline{t},\ \ \overline{y}=\overline{Y},\ \ \overline{u}=\overline{U}-c,\ \ \overline{v}=\overline{V},$$

$$\overline{p}\left(\overline{x},\overline{y}\right)=\overline{P}\left(\overline{X},\overline{Y},\overline{t}\right). \tag{6}$$

Defining the non-dimensional quantities

$$x = \frac{\overline{x}}{\lambda},\ \ y=\frac{\overline{y}}{d_1},\ \ u=\frac{\overline{u}}{c},\ \ v=\frac{\overline{v}}{c\delta},\ \ \delta=\frac{d_1}{\lambda},\ \ h_2=\frac{H_2}{d_2},$$

$$h_1 = \frac{H_1}{d_1},\ \ \phi=\frac{b_1}{a_1},\ \ p=\frac{d_1^2\overline{p}}{c\lambda\mu_0},\ \ S=\frac{d_1}{\mu_0 c}\overline{S}\left(\overline{x}\right),\ \ \mu\left(y\right)=\frac{\overline{\mu}\left(\overline{y}\right)}{\mu_0},$$

$$M = \sqrt{\left(\frac{\sigma}{\mu_0}\right)}B_0 d_1,\ \ Re=\frac{\rho c d_1}{\mu_0},\ \ t=\frac{c\overline{t}}{\lambda},\ \ v=-\frac{\partial\psi}{\partial x},$$

$$u = \frac{\partial\psi}{\partial y},\ \ \gamma_2=\frac{\beta_2 c^2}{\mu_0 d_1^2},\ \ \gamma_3=\frac{\beta_3 c^2}{\mu_0 d_1^2},\ \ Fr=\frac{c^2}{g d_1}. \tag{7}$$

Eqs. (3) and (4) gives

$$Re\delta D_2\psi\frac{\partial\psi}{\partial y} = -M^2\cos^2\theta\left(\frac{\partial\psi}{\partial y}+1\right)\cos\theta -M^2\cos\theta\sin\theta\delta\frac{\partial\psi}{\partial x}-\frac{\partial p}{\partial x} +\delta\frac{\partial S_{xx}}{\partial x}+\frac{\partial S_{xy}}{\partial y}+\frac{Re}{Fr}\sin\gamma, \tag{8}$$

$$Re\delta^2 D_2\psi\frac{\partial\psi}{\partial x} = M^2\cos\theta\sin\theta\left(\frac{\partial\psi}{\partial y}+1\right) M^2\sin^2\theta\delta\frac{\partial\psi}{\partial x} \delta^{-1}\frac{\partial p}{\partial y}-\delta\frac{\partial S_{xy}}{\partial x} -\frac{\partial S_{yy}}{\partial y}-\frac{Re}{Fr}\cos\gamma, \tag{9}$$

where $D_2\psi = \left[\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\right]$. Eqs. (8-9) after invoking long wavelength and small Reynolds number approximation yields

$$\frac{\partial^2 S_{xy}}{\partial y^2}-M^2\cos^2\theta\left(\frac{\partial^2\psi}{\partial y^2}\right)=0, \tag{10}$$

$$\frac{\partial p}{\partial x}=\frac{\partial S_{xy}}{\partial y}-M^2\cos^2\theta\left(\frac{\partial\psi}{\partial y}+1\right)+\frac{Re}{Fr}\sin\gamma, \tag{11}$$

in which

$$S_{xy}=\mu(y)\frac{\partial^2\psi}{\partial y^2}+2\Gamma\left(\frac{\partial^2\psi}{\partial y^2}\right)^3. \tag{12}$$

In order to seek the effect of variable viscosity on peristaltic flow, we let

$$\mu\left(y\right)=1-\alpha\left(y\right) \qquad \alpha<<1, \tag{13}$$

α is the viscosity parameter, Γ is Deborah number

$$\Gamma=\left(\gamma_2+\gamma_3\right). \tag{14}$$

The dimensionless mean flow q is defined as

$$q=F+1+d,$$

in which

$$F=\int_{h_2(x)}^{h_1(x)}\frac{\partial\psi}{\partial y}dy=\psi(h_1)-\psi(h_2).$$

where

$$h_1 = 1+a\cos x,$$
$$h_2 = -d-b\cos\left(x+\phi\right). \tag{15}$$

The boundary conditions in term of stream function ψ are defined as

$$\begin{aligned} \psi &= \frac{F}{2}, \ \frac{\partial\psi}{\partial y}+\beta S_{xy}=-1 \quad \text{at } y=h_1, \\ \psi &= -\frac{F}{2}, \ \frac{\partial\psi}{\partial y}-\beta S_{xy}=-1 \quad \text{at } y=h_2, \end{aligned} \tag{16}$$

where β is a dimensionless slip parameter. The dimensionless pressure rise $\triangle p$ is define by

$$\triangle p=\int_0^1 \frac{dp}{dx}dx, \tag{17}$$

3. PERTURBATION ANALYSIS

In order to solve the present problem, we expanded the flow quantities in a power series of the small Deborah number Γ and small viscosity parameter α

$$\begin{aligned} \psi &= \psi_0+\Gamma\psi_1+... \\ F &= F_0+\Gamma F_1+... \\ p &= p_0+\Gamma p_1+... \end{aligned} \tag{18}$$

$$\begin{aligned} \psi_0 &= \psi_{00}+\alpha\psi_{01}+... \\ \psi_1 &= \psi_{01}+\alpha\psi_{11}+... \\ F_0 &= F_{00}+\alpha F_{01}+.. \\ F_1 &= F_{01}+\alpha F_{11}+... \\ p_0 &= p_{00}+\alpha p_{01}+... \\ p1 &= p_{10}+\alpha p_{11}+.... \end{aligned} \tag{19}$$

If we substitute Eq. (19) into Eqs. (10) to (16) and separate the term of different order in Γ, and α, λ we obtain the following systems of differential equations for the stream function and pressure gradients together with boundary conditions.

CASE-I

$$\frac{dp_{00}}{dx}=\frac{\partial^3\psi_{00}}{\partial y^3}-m^2\left(\frac{\partial\psi_{00}}{\partial y}+1\right)+\frac{Re}{Fr}\sin\gamma, \tag{20}$$

$$\frac{\partial^4\psi_{00}}{\partial y^4}=m^2\frac{\partial^2\psi_{00}}{\partial y^2}, \tag{21}$$

where $m=M\cos\theta$ along with the boundary condition

$$\begin{aligned} \psi_{00} &= \frac{F_{00}}{2}, \ \frac{\partial\psi_{00}}{\partial y}+\beta\frac{\partial^2\psi_{00}}{\partial y^2}=-1 \\ &\quad \text{at } y=h_1, \\ \psi_{00} &= \frac{-F_{00}}{2}, \ \frac{\partial\psi_{00}}{\partial y}-\beta\frac{\partial^2\psi_{00}}{\partial y^2}=-1 \\ &\quad \text{at } y=h_2. \end{aligned} \tag{22}$$

CASE-II

$$\begin{aligned} \frac{dp_{01}}{dx} &= \frac{\partial^3\psi_{01}}{\partial y^3}-m^2\frac{\partial\psi_{01}}{\partial y} \\ &\quad -\frac{\partial}{\partial y}\left(y\frac{\partial^2\psi_{00}}{\partial y^2}\right), \end{aligned} \tag{23}$$

$$\begin{aligned} \frac{\partial^4\psi_{01}}{\partial y^4} &= m^2\frac{\partial^2\psi_{01}}{\partial y^2} \\ &\quad +\frac{\partial^2}{\partial y^2}\left(y\frac{\partial^2\psi_{00}}{\partial y^2}\right), \end{aligned} \tag{24}$$

with the corresponding boundary conditions

$$\begin{aligned} \text{at } y=h_1: \psi_{01} &= \frac{F_{01}}{2}, \\ \frac{\partial\psi_{01}}{\partial y}+\beta\frac{\partial^2\psi_{01}}{\partial y^2}-\beta y\frac{\partial^2\psi_{00}}{\partial y^2} &= 0, \\ \text{at } y=h_2: \psi_{01} &= \frac{-F_{01}}{2} \\ ,\ \frac{\partial\psi_{01}}{\partial y}-\beta\frac{\partial^2\psi_{01}}{\partial y^2}+\beta y\frac{\partial^2\psi_{00}}{\partial y^2} &= 0. \end{aligned} \tag{25}$$

CASE-III

$$\frac{dp_{10}}{dx}=\frac{\partial^3\psi_{10}}{\partial y^3}-M^2\cos^2\theta\frac{\partial\psi_{10}}{\partial y}+2\frac{\partial}{\partial y}\left(y\frac{\partial^2\psi_{00}}{\partial y^2}\right)^3 \tag{26}$$

$$M^2\cos^2\theta\frac{\partial^2\psi_{10}}{\partial y^2}+\frac{\partial^4\psi_{10}}{\partial y^4}=6\frac{\partial}{\partial y}\left(\left(\frac{\partial^2\psi_{00}}{\partial y^2}\right)^2\frac{\partial^3\psi_{00}}{\partial y^3}\right) \tag{27}$$

subject to boundary conditions

$$\begin{aligned} \text{at } y=h_1: \psi_{10} &= \frac{F_{10}}{2} \\ ,\ \frac{\partial\psi_{10}}{\partial y}+\beta\frac{\partial^2\psi_{10}}{\partial y^2} &= -2\beta\left(\frac{\partial^2\psi_{00}}{\partial y^2}\right)^3 \\ \text{at } y=h_2: \psi_{10} &= \frac{-F_{10}}{2} \\ ,\ \frac{\partial\psi_{10}}{\partial y}-\beta\frac{\partial^2\psi_{10}}{\partial y^2} &= 2\beta\left(\frac{\partial^2\psi_{00}}{\partial y^2}\right)^3. \end{aligned} \tag{28}$$

3.1 Series solution of the Problem .

The approximate solution is given as

$$\begin{aligned}
\psi \;=\;& (y-h_1)\left(\frac{F_{00}}{(h_1-h_2)}-\frac{(F_{00}+(h_1-h_2))\,S_{11}}{L_2\,(h_1-h_2)}\right)\\
&+\frac{(F_{00}+(h_1-h_2))\,m\,(\cosh[m(y-h_2)]-\cosh[m(y-h_1)])}{L_2}\\
&+\frac{(F_{00}+(h_1-h_2))\,m\,(1-\cosh[m(h_1-h_2)])}{L_2}\\
&+\frac{(F_{00}+(h_1-h_2))\,\beta m^2\,(\sinh[m(y-h_1)]+\sinh[m(y-h_2)])}{L_2}\\
&-\frac{(F_{00}+(h_1-h_2))\,\beta m^2\sinh[m(h_1-h_2)]}{L_2}\\
&+\alpha\left[\frac{F_{01}}{2}+(y-h_1)\left(\frac{F_{01}-S_3}{(h_1-h_2)}+\frac{S_{10}\,(\cosh[mh_2]-\cosh[mh_1])}{L\,(h_1-h_2)}\right)\right.\\
&+\frac{(y-h_1)\,S_{11}}{L_2}\left(-F_{01}+S_3+S_8-\frac{S_{10}S_{12}}{L}\right)\\
&+\frac{S_{10}}{L}\,(\cosh[my]-\cosh[mh_1])+\frac{(h_1-h_2)\,(S_3+S_8)}{L_2}\\
&-\frac{(h_1-h_2)\,F_{01}}{L_2}-\frac{S_{10}\,(h_1-h_2)}{LL_2}S_{12}\;m\,(\cosh[m(y-h_2)]-\cosh[m(y-h_1)])\\
&-\frac{S_{10}\,(h_1-h_2)}{LL_2}S_{12}\;m\,(1-\cosh[m(h_1-h_2)])\\
&-\frac{S_{10}\,(h1-h2)}{LL_2}S_{12}\;\beta m^2\sinh[m(y-h_1)]+\sinh[m(y-h_2)]\\
&+\frac{S_{10}\,(h_1-h_2)}{LL_2}S_{12}\;\beta m^2\sinh[m(h_1-h_2)]\\
&+\Gamma\left[\frac{F_{10}}{2}+(y-h_1)\left(\frac{F_{10}-K_3}{(h_1-h_2)}+\frac{K_{10}\,(\cosh[mh_2]-\cosh[mh_1])}{L\,(h_1-h_2)}\right)\right.\\
&+\frac{(y-h_1)\,S_{11}}{L_2}\left(-F_{10}+K_3+K_8-\frac{K_{10}S_{12}}{L}\right)\\
&+\frac{K_{10}}{L}\,(\cosh[my]-\cosh[mh_1])+\frac{(h_1-h_2)\,(K_3+K_8)}{L_2}-\frac{(h_1-h_2)\,F_{10}}{L_2}\\
&-\frac{K_{10}\,(h1-h2)}{LL_2}S_{12}\;m\,(\cosh[m(y-h_2)]-\cosh[m(y-h_1)])\,.
\end{aligned}$$

$$\begin{aligned}
\frac{dp}{dx} \;=\;& Re\frac{\sin\gamma}{Fr}-m^2-m^2\left(\frac{F}{(h_1-h_2)}-\frac{(F+(h_1-h_2))\,S_{11}}{L2\,(h_1-h_2)}\right)\\
&-\alpha\left[\begin{array}{c}
\frac{S10(\cosh[mh2]-\cosh[mh1])}{L(h_1-h_2)}-\frac{S3}{h1-h2}+\frac{S_{11}}{L_2}\,(S_3+S_8)\\
-\frac{S_{11}}{L_2}\frac{S10}{L(h_1-h_2)}\,(\cosh[mh2]-\cosh[mh1]+m\sinh[mh1])\\
-\frac{S_{11}}{L_2}\frac{S10}{L(h_1-h_2)}\beta m^2\cosh[mh1]\,(h_1-h_2)
\end{array}\right]\\
&-\Gamma\left[\begin{array}{c}
\frac{K10(\cosh[mh2]-\cosh[mh1])}{L(h_1-h_2)}-\frac{K3}{(h_1-h_2)}+\frac{S_{11}}{L2}\,(K_3+K_8)\\
-\frac{K_{11}}{L_2}\frac{K10}{L(h_1-h_2)}\,(\cosh[mh2]-\cosh[mh1]+m\sinh[mh1])\\
+\frac{K_{11}}{L_2}\frac{K10}{L(h_1-h_2)}\beta m^2\cosh[mh1]\,(h_1-h_2)
\end{array}\right].
\end{aligned} \tag{29}$$

4. NUMERICAL SOLUTION

The complex behaviour of the non-newtonian fluids can be dealt more swiftly with the help of numerical solvers such as discussed by (Sohail *et al.* 2014; Rashidi *et al.* 2012) and many more. In this paper, Eqs. (10) and (11) subject to the boundary conditions, Eqs. (16) were solved numerically using a finite difference code. This technique implements a collocation formula, the collocation polynomial provides a continuous solution which is fourth order accurate homogeneously in the preferred domain. The nonlinear equations are solved iteratively by linearization, therefore our numerical approach relies upon the linear equation solvers of Matlab.

5. GRAPHS AND DISCUSSION

The main aim of this section is to study the behavior of involved key parameters on pressure rise per wave length Δp, axial velocity u and trapping. Therefore we have prepared Figs. 2 to 11. Figs. 2a and b represent the comparison of pressure rise for both perturbation and numerical solutions. It is observed from Fig. 2a that the perturbation and numerical solutions are almost identical in the pumping region while in Fig 2b there is a difference between the perturbation and numerical solution. The comparison between perturbation and numerical solutions on axial velocity are discussed in Figs. 2c and d (bottom row, left to right). Fig dc shows a difference between perturbation and numerical solution. It is observed from Fig. 2d that there is a good agreement between numerical and perturbation solution near the walls of the channel.

The variation of pressure rise with q is studied in Figures 3, 4 and 5, for various values of Γ, α, β and M, (Figure 3) θ, ϕ, γ and Fr (Figure 4), b and a (Figure 5).

At lower Deborah numbers, the fluid flow is associated with Newtonian viscous flow. At higher Deborah numbers ($\Gamma > 0$), the material behaves like non-Newtonian fluid, increasingly dominated by elasticity. Fig. 3a studies the effect of Γ on Δp. The solid line curve is for Newtonian fluid ($\Gamma = 0$). It is found that the pumping curve for third order fluid lies below the Newtonian fluid in the pumping region, while in free pumping there is no difference in Newtonian and third order fluid. Fig. 3b and 3c show the effect of α and β on Δp. It is noted that magnitude of pressure rise decreases by increases α and β. In Fig. 3d we describe the influence of M. It reveal that pumping rate decreases by increasing M.

Fig. 4a is the graph of Δp for different values of θ. It indicates that the pumping rate increases with an increase in θ for both free pumping and copumping regions. To observe the effect of phase difference ϕ on Δp, we have presented in Fig. 4b. It shows that in pumping region ($\Delta p > 0$), the pumping increases with an increase in ϕ. However in copumping and free pumping regions this situation is quite opposite. Fig 4c describes the influence of angle of inclination γ. It shows that the pressure rise increases in all regions with an increase in γ. Fig. 4d depicts the variation of Δp with Fr. It is noticed that with an increase in Froude number, the pumping rate decreases in all regions.

The observations regarding the effects of lower and upper wave amplitudes b and a respectively on Δp are quite opposite to that as in the case of ϕ Figs. 5a and 5b.

Fig. 6a studies the effect of Γ on the velocity u versus y. We observe that there is no effect of Γ on the velocity at the centre of the channel as the curves coincide. Fig. 6b represents the graph of the velocity u for three values of α. It is clear from the figure that the velocity increases by increasing α. The variation of u with y for different values of β and M are studied in Fig. 7a and Fig. 7b. It is obvious from the figures that increasing β and M led to decrease in the velocity.

6. CONCLUSIONS AND FUTURE WORK

This paper presented a detailed analysis of the consequence of slip conditions and crucial fluid parameters including the variable viscosity and inclined magnetic field, that effect the peristaltic transport of a third order fluid in an inclined asymmetric channel. Detailed understanding of these flow parameters can help to understand the complex rheology of biological fluids such as the peristaltic mechanism involved in lymphatic vessels, narrow blood vessels and intestine. The perturbation theory has been applied to approximate the solution of Eqs. (10) and (11) with boundary conditions (16). For the confirmation of the perturbation solution, we presented a graphical analysis of the variation in fluid velocity and pressure gradient relative to dimensionless mean flow, based on a collection of judiciously selected parametric values. The main results can be summarized as: (a) The pressure rise increases in all regions with an increase in γ (which is the angle at which the channel is inclined to the horizontal), while it decreases in all regions with increasing Fr (which is the ratio of the charac-

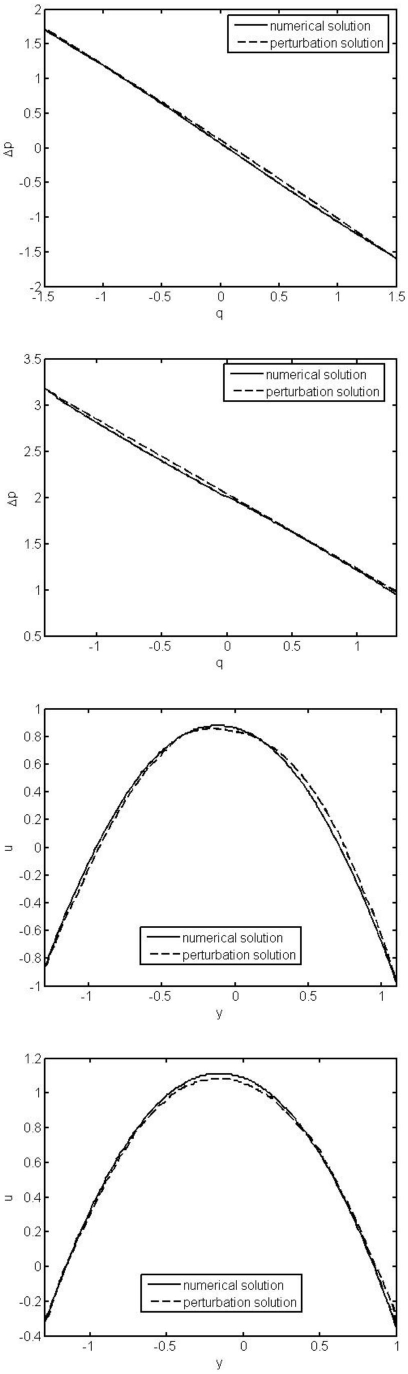

Fig. 2. (a) Comparison of $\triangle p$ for fixed Γ=0.02, ϕ=0.3, α=0.2, M=1.5, $\theta=\frac{\pi}{4}$, $Fr=1$, $\beta=0.4$, $\gamma=\frac{\pi}{6}$, $b=0.7$, $a=0.4$, $d=1$, $Re=4$.; (b) Comparison of $\triangle p$ for fixed M=1.5, Γ=0.01, ϕ=0.2, $\beta=0.4$, $\theta=\frac{\pi}{6}$, $Fr=1$, $\alpha=0.2$, $\gamma=\frac{\pi}{6}$, $b=0.7$, $a=0.4$, $d=1$, $Re=5$.; (c) Comparison of u for fixed $\Gamma=0.02$, $\phi=0.4$, $\alpha=0.4$, $M=1.5$, $\theta=\frac{\pi}{6}$, $Fr=1$, $\beta=0.5$, $b=0.4$, $a=0.4$, $d=1$, $q=2$.; (d) Comparison of u for fixed $\Gamma=0.01$, $\phi=0.4$, $\alpha=0.2$, $M=0.5$, $\theta=\frac{\pi}{6}$, $Fr=1$, $\beta=0.5$, $b=0.4$, $a=0.4$, $d=1$, $q=4$.;

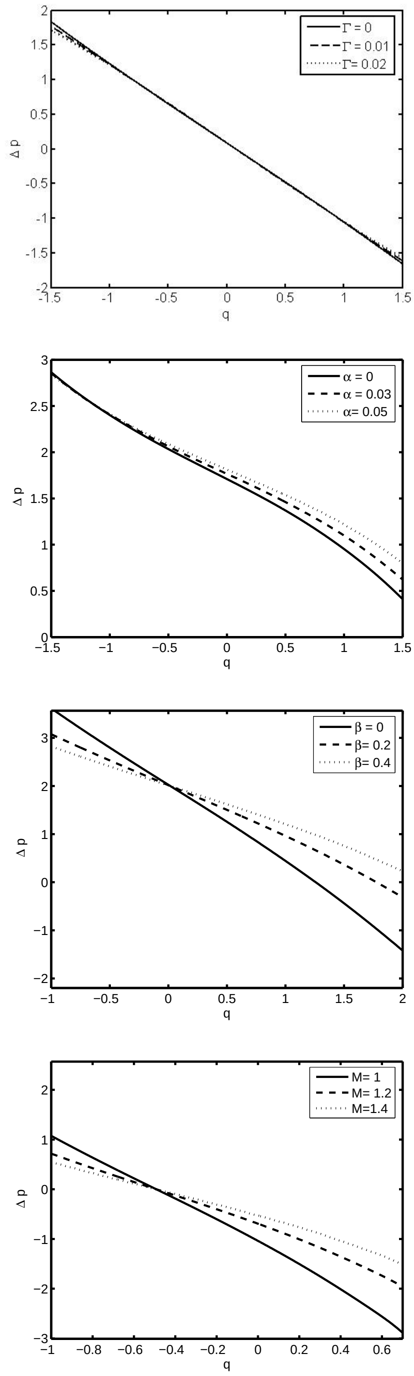

Fig. 3. (a) Variation of Γ on $\triangle p$ when $\phi=0.3$, $\alpha=0.2$, $M=1.5$, $\theta=\frac{\pi}{4}$, $Fr=1$, $\beta=0.4$, $\gamma=\frac{\pi}{6}$, $b=0.7$, $a=0.4$, $d=1$, $Re=4$.; (b) Variation of α on $\triangle p$ when $\phi=0.1$, $\Gamma=0.01$, $M=0.5$, $\theta=\frac{\pi}{4}$, $Fr=1$, $\beta=0.4$, $\gamma=\frac{\pi}{6}$, $b=0.7$, $a=0.4$, $d=1$, $Re=5$.; (c) Variation of β on $\triangle p$ when $\Gamma=0.01$, $\phi=0.2$, $M=1.5$, $\theta=\frac{\pi}{6}$, $Fr=1$, $\alpha=0.2$, $\gamma=\frac{\pi}{6}$, $b=0.7$, $a=0.4$ $d=1$, $Re=5$.; (d) Variation of M on $\triangle p$ when $\Gamma=0.01$, $\phi=0.1$, $\beta=0.1$, $\theta=\frac{\pi}{6}$, $Fr=1$, $\alpha=0.1$, $\gamma=\frac{\pi}{6}$, $b=0.7$, $a=0.4$ $d=1$, $Re=5$.

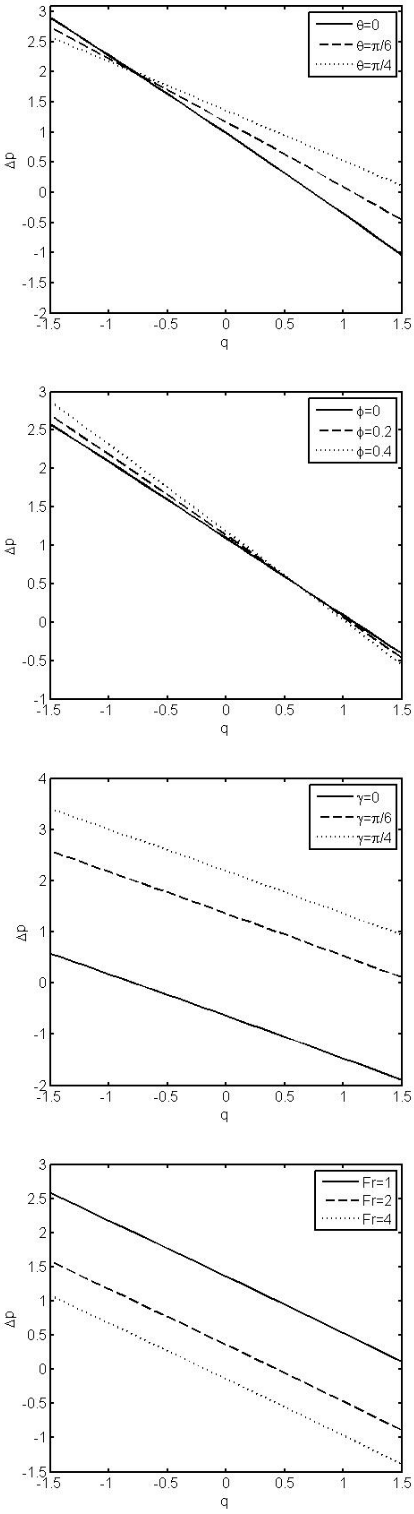

Fig. 4. (a) Variation of θ on $\triangle p$ when $\Gamma = 0.01$, $\phi = 0.3$, $M = 1.5$, $\alpha = 0.2$, $Fr = 1$, $\beta = 0.4$, $\theta = \frac{\pi}{6}$, $\gamma = \frac{\pi}{6}$, $b = 0.7$, $a = 0.4$ $d = 1$, $Re = 4$.; (b) Variation of ϕ on $\triangle p$ when $\Gamma = 0.01$, $\alpha = 0.1$, $M = 0.5$, $\theta = \frac{\pi}{6}$, $Fr = 1$, $\beta = 0.4$, $\gamma = \frac{\pi}{4}$, $b = 0.7$, $a = 0.4$ $d = 1$, $Re = 4$.; (c) Variation of γ on $\triangle p$ when $\Gamma = 0.01$, $\alpha = 0.1$, $M = 0.5$, $\theta = \frac{\pi}{6}$, $Fr = 1$, $\beta = 0.4$, $\alpha = 0.2$ $b = 0.7$, $a = 0.4$ $d = 1$, $Re = 5$.; (d) Variation of Fr on $\triangle p$ when $\phi = 0.3$, $\Gamma = 0.01$, $M = 1.5$, $\alpha = 0.2$ $\theta = \frac{\pi}{4}$, $\beta = 0.4$, $\gamma = \frac{\pi}{6}$, $b = 0.7$, $a = 0.4$ $d = 1$, $Re = 4$.

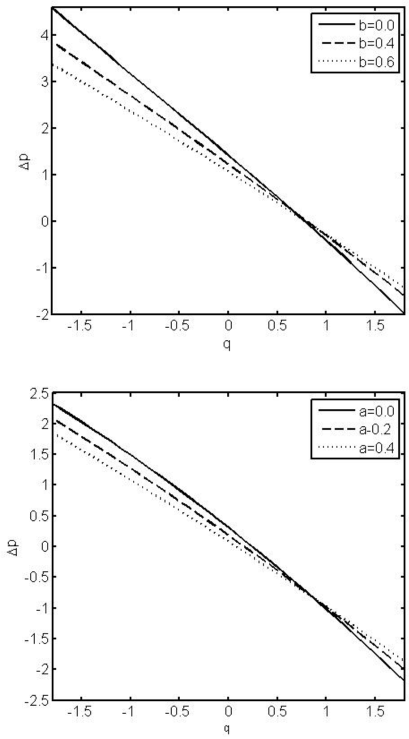

Fig. 5. Left to right, (a) Fig 1k Variation of b on $\triangle p$ when $\Gamma = 0.02$, $\phi = 0.3$, $M = 1.5$, $\alpha = 0.2$ $\theta = \frac{\pi}{6}$, $Fr = 1$ $\beta = 0.3$, $\gamma = \frac{\pi}{6}$, $a = 0.4$ $d = 1$, $Re = 4$; (b) Fig 1l Variation of a on $\triangle p$ when $\Gamma = 0.02$, $\phi = 0.3$, $M = 1.5$, $\alpha = 0.1$ $\theta = \frac{\pi}{6}$, $Fr = 1$ $\beta = 0.4$, $\gamma = \frac{\pi}{3}$, $b = 0.7$ $d = 1$, $Re = 1$.

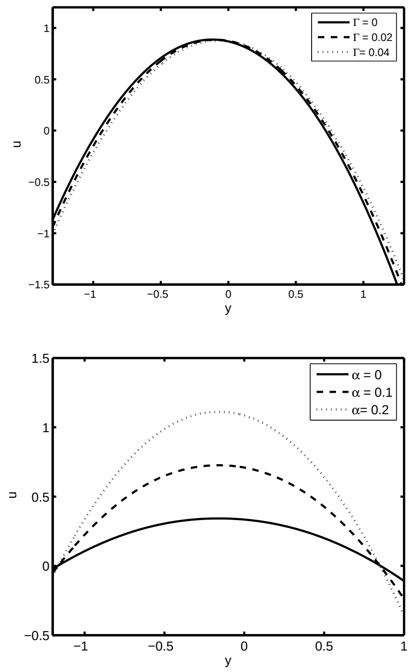

Fig. 6. From left to right (a) Variation of Γ on u when $\phi = 0.4$, $M = 0.5$, $\theta = \frac{\pi}{6}$, $\beta = 0.5$, $q = 2$, $a = 0.4$, $b = 0.4$, $d = 1$; (b) Variation of α on u when $\Gamma = 0.01$, $\phi = 0.4$, $M = 0.5$, $\theta = \frac{\pi}{6}$, $\beta = 0.5$, $q = 2$, $a = 0.4$, $b = 0.4$, $d = 1$.;

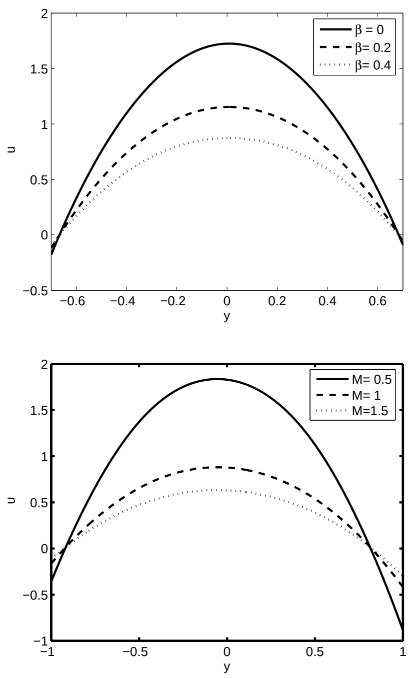

Fig. 7. (a) Variation of β on u when $\Gamma = 0.01$ $\phi = 0.4, M = 0.5, \theta = \frac{\pi}{6}, \alpha = 0.2,$ $q = 2,\ a = 0.4, b = 0.4,\ d=1.$; (b) Variation of M on u when $\phi = 0.4,\ \Gamma = 0.01,\ \theta = \frac{\pi}{6},$ $\beta = 0.5, q = 2,\ a = 0.4, b = 0.4,\ d = 1.$

teristic velocity to a gravitational wave velocity). This shows that an increased inclination of channel increases the pressure, whereas increased characteristic velocity will reduce the pressure. (b) The axial velocity increases with increasing θ, ϕ and α, while it decreases with increasing β and M (i.e. the Hartman number which is the ratio of magnetic effect to viscous effect). We conclude from this contrasting velocity profile under the influence of flow parameters that the fluid flow is influenced by the inclination, the slip condition and the electromagnetic parameter. Our future work will take into account the thermal effects which have great importance in the peristalsis, such as heat conduction in biological tissues, heat convection due to the blood flow through the pores of the tissues, and radiation between surface and its environment. Many researchers studied the peristalsis of viscous and non-Newtonian fluids with and without heat transfer in the peristaltic flow phenomenon. Recently these effects in both Newtonian and viscoelastic fluid flows have been investigated. The peristaltic flow of a third order fluid in an inclined asymmetric channel, with slip condition, variable viscosity, heat transfer and inclined magnetic field will be a challenging problem and will be analyzed both analytically and numerically in our future work.

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APPENDIX

$$
\begin{aligned}
m &= M\cos\theta, \quad L_1 = m\left(\sinh[mh_1]-\sinh[mh_2]\right)+\beta m^2\left(\cosh[mh_1]+\cosh[mh_2]\right), \\
L_2 &= 2m\left(1-\cosh[m(h_1-h_2)]\right)-2\beta m^2\sinh[m(h_1-h_2)], \\
&+(h_1-h_2)\left(m^2\sinh[m(h_1-h_2)]\right)+(h_1-h_2)2\beta m^3\cosh[m(h_1-h_2)] \\
&+(h_1-h_2)\beta^2 m^4\sinh[m(h_1-h_2)], \\
P_1 &= \frac{(F+(h_1-h_2))\,m\left(\cosh[mh_1]-\cosh[mh_2]\right)}{L_2} \\
&+\frac{\beta m^2\left(\sinh[mh_1]-\sinh[mh_2]\right)\times(F+(h_1-h_2))}{L_2}, \\
P_2 &= \frac{(F+(h_1-h_2))\,m\left(\sinh[mh_1]-\sinh[mh_2]\right)}{L_2} \\
&+\frac{\beta m^2\left(\cosh[mh_1]-\cosh[mh_2]\right)\times(F+(h_1-h_2))}{L_2}, \\
P_3 &= P_2^2+2P_1^2P_2, \quad P_4=P_1^2+2P_1P_2^2, \quad P_5=P_2P_1^2, \quad P_6=P_1P_2^2, \\
S_1 &= \frac{3P_2h_1\sinh[mh_1]}{2}-\frac{3P_1h_1\cosh[mh_1]}{2}+\frac{P_1mh_1^2\sinh[mh_1]}{2}-\frac{P_2mh_1^2\cosh[mh_1]}{2}, \\
S_2 &= \frac{3P_2h_2\sinh[mh_2]}{2}-\frac{3P_1h_2\cosh[mh_2]}{2}+\frac{P_1mh_2^2\sinh[mh_2]}{2}-\frac{P_2mh_2^2\cosh[mh_2]}{2}
\end{aligned}
$$

$$
\begin{aligned}
S_3 &= P_1\frac{(-3h_1\cosh[mh_1]+3h_2\cosh[mh_2])}{2}+P_1m\left(\frac{h_1^2\sinh[mh_1]-h_2^2\sinh[mh_2]}{2}\right) \\
&+P_2\left(\frac{3h_2\sinh[mh_1]-3h_1\sinh[mh_2]}{2}\right)-P_2m\left(\frac{h_1^2\sinh[mh_1]-h_2^2\cosh[mh_2]}{2}\right),
\end{aligned}
$$

$$
\begin{aligned}
S_4 &= \frac{3P_1\cosh[mh_1]}{2}-\frac{h_1P_1\sinh[mh_1]}{2}+\frac{3P_2\sinh[mh_1]}{2} \\
+ &\quad \frac{P_2mh_1\cosh[mh_1]}{2}+\frac{P_1m^2h_1^2}{2}\cosh[mh1]-\frac{P_2m^2h_1^2}{2}\sinh[mh_1]
\end{aligned}
$$

$$
\begin{aligned}
S_5 &= \frac{3P_1\cosh[mh_2]}{2}-\frac{h_2P_1\sinh[mh_2]}{2}+\frac{3P_2\sinh[mh_2]}{2}+ \\
&\quad \frac{P_2mh_2\cosh[mh_2]}{2}+\frac{P_1m^2h_2^2\cosh[mh_2]}{2}-\frac{P_2m^2h_2^2\sinh[mh_2]}{2},
\end{aligned}
$$

$$
\begin{aligned}
S_6 &= -2mP_1\sinh[mh_1]+\frac{P1m^2h_1\cosh[mh_1]}{2}+2P_2m\cosh[mh_1]- \\
&\quad \frac{P_2m^2h_1\sinh[mh_1]}{2}+\frac{P_1m^3h_1^2\sinh[mh_1]}{2}-\frac{P_2m^3\,h_1^2\cosh[mh1]}{2},
\end{aligned}
$$

$$S_7 = -2mP_1\sinh[mh_2] + \frac{P_1 m^2 h_2 \cosh[mh_2]}{2} + 2P_2 m\cosh[mh_2]$$
$$-\frac{P_2 m^2 h_2 \sinh[mh_2]}{2} + \frac{P_1 m^3 h_2^2 \sinh[mh_2]}{2} - \frac{P_2 m^3 h_2^2 \cosh[mh_2]}{2},$$
$$S_8 = -(-\beta R_1 + \beta S_6 + S_4), \quad S_9 = -(\beta R_2 - \beta S_7 + S_5), \quad S_{10} = S_8 - S_9,$$
$$S_{11} = \left(2m(1-\cosh[m(h_1-h_2)]) - 2\beta m^2 \sinh[m(h_1-h_2)]\right),$$
$$S_{12} = \frac{\cosh[mh_2]-\cosh[mh_1]}{(h_1-h_2)} + m\sinh[mh_1] + \beta m^2\cosh[mh_1],$$

$$R_1 = h_1 m^2 P_1 \cosh[mh_1] - h_1 P_2 m^2 \sinh[mh_1],$$
$$R_2 = h_2 m^2 P_1 \cosh[mh_2] - h_2 P_2 m^2 \sinh[mh_2],$$
$$R_4 = m^6 (\cosh[mh_1])^3 P_1^3 - (\sinh[mh_1])^3 P_2^3$$
$$-3P_2 \sinh[mh_1]^6 (\cosh[mh_1])^2 P_1^2 + 3P_1 \cosh[mh_1] P_2^2 (\sinh[mh_1])^2,$$
$$R_5 = m^6 (\cosh[mh_2])^3 P_1^3 - (\sinh[mh2h_2])^3 P_2^3$$
$$-3P_2 \sinh[mh_2]^6 (\cosh[mh_2])^2 P_1^2 + 3P_1 \cosh[mh_2] P_2^2 (\sinh[mh_2])^2,$$

$$K_1 = \frac{m^4 P_3 [\sinh[3mh_1] - 12mh_1 \cosh[mh_1]]}{8} - \frac{m^4 P_4 [\cosh[3mh_1] + 12mh_1 \sinh[mh_1]]}{8}$$
$$+m^4 P_5 \left[\frac{\sinh[3mh_1]}{8} + \frac{9mh_1 \cosh[mh_1]}{2}\right] + m^4 P_6 \left[\cosh\frac{[3mh_1]}{8} + \frac{9mh_1 \sinh[mh_1]}{2}\right],$$

$$K_2 = \frac{m^4 P_3 [\sinh[3mh_2] - 12mh_2 \cosh[mh_2]]}{8} - \frac{m^4 P_4 [\cosh[3mh_2] + 12mh_2 \sinh[mh_2]]}{8}$$
$$+m^4 P_5 \left[\frac{\sinh[3mh_2]}{8} + \frac{9mh_2 \cosh[mh_2]}{2}\right] + m^4 P_6 \left[\cosh\frac{[3mh_2]}{8} + \frac{9mh_2 \sinh[mh_2]}{2}\right],$$

$$K_3 = \frac{m^4 P_3 [\sinh[3mh_1] - \sinh[3mh_2]]}{8} - \frac{m^4 P_4 [\cosh[3mh_1] - \cosh[3mh_2]]}{8}$$
$$-\frac{m^4 P_3 12m (h_1 \cosh[mh_1] - h_2 \cosh[mh_2])}{8}$$
$$+\frac{m^4 P_4 12m (h_1 \sinh[mh_1] - h_2 \sinh[mh_2])}{8} + \frac{m^4 P_5 [3mh_1] - \sinh[3mh_2]}{8}$$
$$+\frac{m^4 P_5 9m (h_1 \cosh[mh_1] - h_2 \cosh[mh_2])}{2} + \frac{\cosh[3mh_1] - \cosh[3mh_2]}{8}$$
$$+m^4 P_6 \frac{9m (h_1 \sinh[mh_1] - h_2 \sinh[mh_2])}{2},$$

$$K_4 = \frac{m^4 P_3 \left[3m\cosh[3mh_1] - 12m\cosh[mh_1] - 12m^2 h_1 \sinh[mh_1]\right]}{8}$$
$$-\frac{m^4 P_4 \left[3m\sinh[3mh_1] - 12m\sinh[mh_1] - 12m^2 h_1 \cosh[mh_1]\right]}{8}$$
$$+P_5 m^4 \left[\frac{3m\cosh[3mh_1]}{8} + \frac{9m\cosh[mh_1]}{2} + \frac{9h_1 m^2 \sinh[mh_1]}{2}\right]$$
$$-m^4 P_6 \left[\frac{3m\sinh[3mh_1]}{8} + \frac{9m\sinh[mh_1]}{2} + \frac{9h_1 m^2 \cosh[mh_1]}{2}\right]$$

$$
\begin{aligned}
K_5 \quad = \quad & \frac{m^4 P_3 \left[3m\cosh[3mh_2] - 12m\cosh[mh_2] - 12m^2 h_2 \sinh[mh_2]\right]}{8} \\
& - \frac{m^4 P_4 \left[3m\sinh[3mh_2] - 12m\sinh[mh_2] - 12m^2 h_2 \cosh[mh_2]\right]}{8} \\
& + P_5 m^4 \left[\frac{3m\cosh[3mh_2]}{8} + \frac{9m\cosh[mh_2]}{2} + \frac{9h_2 m^2 \sinh[mh_2]}{2}\right] \\
& - m^4 P_6 \left[\frac{3m\sinh[3mh_2]}{8} + \frac{9m\sinh[mh_2]}{2} + \frac{9h_2 m^2 \cosh[mh_2]}{2}\right]
\end{aligned}
$$

$$
\begin{aligned}
K_6 \quad = \quad & \frac{P_3 m^4 \left[9m^2\sinh[3mh_1] - 24m^2\sinh[mh_1] + 12m^3 h_1 \cosh[3mh_1]\right]}{8} \\
& - \frac{mP_4 \left[9m^2\cosh[3mh_1] + 24m^2\cosh[mh_1] + 12m^3 h_1 \sinh[3mh_1]\right]}{8} + \\
& P_5 m^4 \left[\frac{9m^2\sinh[3mh_1]}{8} + 9m^2\sinh[mh_1] + \frac{9m^3\cosh[mh_1]}{2}\right] \\
& - m^4 P_6 \left[\frac{9m^2\cosh[3mh_1]}{8} + 9m^2\cosh[mh_1] + \frac{9m^3\sinh[mh_1]}{2}\right]
\end{aligned}
$$

$$
\begin{aligned}
K_7 \quad = \quad & \frac{P_3 m^4 \left[9m^2\sinh[3mh_2] - 24m^2\sinh[mh_2] + 12m^3 h_2 \cosh[3mh_2]\right]}{8} \\
& - \frac{m^4 P_4 \left[9m^2\cosh[3mh_2] + 24m^2\cosh[mh_2] + 12m^3 h_2 \sinh[3mh_2]\right]}{8} \\
& P_5 m^4 \left[\frac{9m^2\sinh[3mh_2]}{8} + 9m^2\sinh[mh_2] + \frac{9m^3\cosh[mh_2]}{2}\right] \\
& - m^4 P_6 \left[\frac{9m^2\cosh[3mh_2]}{8} + 9m^2\cosh[mh_2] + \frac{9m^3\sinh[mh_2]}{2}\right], \\
K_8 \quad = \quad & -(K_4 + \beta K_6 + 2\beta R_4), \quad K_9 = -(K_5 - \beta K_7 - 2\beta R_5), \quad K_{10} = K_8 - K_7,
\end{aligned}
$$