Journal of Applied Fluid Mechanics, Vol. 9, No. 4, pp. 2025-2036, 2016.
Available online at www.jafmonline.net, ISSN 1735-3572, EISSN 1735-3645.
DOI: 10.18869/acadpub.jafm.68.235.24834

Boundary Layer Flow and Heat Transfer over a Permeable Exponentially Stretching/Shrinking Sheet with Generalized Slip Velocity

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(Received March, 28, 2015; accepted October, 7, 2015)

Abstract

In this paper, the steady laminar boundary layer flow and heat transfer over a permeable exponentially stretching/shrinking sheet with generalized slip velocity is studied. The flow and heat transfer induced by stretching/shrinking sheets are important in the study of extrusion processes and is a subject of considerable interest in the contemporary literature. Appropriate similarity variables are used to transform the governing nonlinear partial differential equations to a system of nonlinear ordinary (similarity) differential equations. The transformed equations are then solved numerically using the bvp4c function in MATLAB. Dual (upper and lower branch) solutions are found for a certain range of the suction and stretching/shrinking parameters. Stability analysis is performed to determine which solutions are stable and physically realizable and which are not stable. The effects of suction parameter, stretching/shrinking parameter, velocity slip parameter, critical shear rate and Prandtl number on the skin friction and heat transfer coefficients as well as the velocity and temperature profiles are presented and discussed in detail. It is found that the introduction of the generalized slip boundary condition resulted in the reduction of the local skin friction coefficient and local Nusselt number. Finally, it is concluded from the stability analysis that the first (upper branch) solution is stable while the second (lower branch) solution is not stable.

Keywords: Boundary layer; Heat transfer; General slip; Stretching/shrinking; Numerical solution; Dual solutions; Stability analysis.

Nomenclature

A, B	constants	T	fluid temperature
a, b	constants	T_0	constant
C_f	skin friction coefficient, Eq. (16)	T_∞	ambient temperature
c_p	specific heat at constant pressure	U_0	constant velocity characteristic of the sheet
e	exponent		
$F(\eta)$	small relative to stream function	u, v	velocity components along x and y axes
$f(\eta)$	dimensionless stream function	u_t	tangential sheet velocity
$G(\eta)$	small relative to temperature function	v_0	constant mass flux velocity
k	fluid thermal conductivity	w	condition on the surface
L	characteristic length of the sheet	x, y	Cartesian coordinates
Nu_x	local Nusselt number, Eq. (16)		
Pr	Prandtl number, Eq. (12)	$\alpha(x)$	velocity slip parameter, Eq. (11)
q_w	surface heat flux	$\alpha*$	Navier's constant slip length, Eq. (12)
Re_x	local Reynolds number, Eq. (18)	$\beta(x)$	critical shear rate, Eq. (11)
s	mass flux parameter, Eq. (8)	$\beta*$	reciprocal of critical shear rate, Eq. (12)

η	independent similarity variable
γ	unknown eigenvalue parameter, Eq. (23)
λ	stretching/shrinking parameter, Eq. (5)
μ	dynamic viscosity
ν	kinematic viscosity
ψ	stream function
ρ	fluid density
τ	dimensionless time variable, Eq. (19)
τ_w	shear stress along the surface
$\theta(\eta)$	dimensionless temperature function

1. Introduction

The study of viscous flow past a stretching surface has enormous applications in technological and engineering processes, such as wire drawing, roofing shingles, paper production and others. Sakiadis (1961) was the first to consider the problem of boundary layer flow over a stretching sheet, which was verified experimentally by Tsou *et al.* (1967), then followed by Crane (1970), who extended the idea for the two-dimensional problem. The uniqueness of the solution obtained in (Crane 1970) was investigated by McLeod and Rajagopal (1987). Further, Gupta and Gupta (1977) and Magyari and Keller (2000) studied the heat and mass transfer over a stretching sheet subject to suction or blowing. Later, Nazar *et al.* (2004) considered the unsteady boundary layer flow due to a stretching surface in a rotating fluid, while Ishak *et al.* (2008) investigated the heat transfer over a stretching surface with uniform or variable heat flux in micropolar fluids.

Recently, problems involving shrinking sheets become significantly important in the industry, where the fluid flow is shrunk towards the origin of the surface. The study of such flows was first performed by Wang (1990). Later, Miklavcic and Wang (2006) proved the existence of the dual solutions for steady hydrodynamic flow due to a permeable shrinking sheet for a certain value of the suction parameter. Since then, numerous studies related to fluid flow over a shrinking sheet are conducted for different physical properties (see Fang *et al.* (2009), Hayat *et al.* (2009), Bachok *et al.* (2010), Bhattacharyya and Layek (2011), Rohni *et al.* (2012), Ali *et al.* (2013), Saleh *et al.* (2014), among others). It is worth mentioning to this end that this new type of shrinking sheet flow is essentially a backward flow as discussed by Goldstein (2006) and it shows physically phenomena quite distinct from the stretching flow.

Over the last few decades, most of the studies conducted were about the linear or nonlinear stretching/shrinking flat sheets, while little attention has been paid to the study of boundary layer flow over an exponentially stretching/shrinking sheet. It seems that Magyari and Keller (1999) were the first to investigate the flow over an exponentially stretching continuous surface. On the other hand, Elbashbeshy (2001) studied the heat transfer over an exponentially stretching continuous surface by considering suction, while Sanjayanand and Khan (2006) and Khan (2006) investigated the viscous-elastic boundary layer flow over an exponentially stretching sheet. Bhattacharyya (2011) studied the flow and heat transfer over an exponentially shrinking sheet and found that a steady flow is possible when the mass suction parameter exceeds a certain critical value. Later, Bhattacharyya and Vajravelu (2012) investigated the stagnation point flow and heat transfer over an exponentially shrinking sheet, while Bachok *et al.* (2012) investigated the steady two-dimensional stagnation-point flow of a water-based nanofluid over an exponentially stretching/shrinking sheet. Further, the study of steady laminar two-dimensional flow and heat transfer over an exponentially shrinking vertical sheet with suction has been studied by Rohni *et al.* (2013), while Kasmuri *et al.* (2013) studied the boundary layer stagnation-point flow and heat transfer past a permeable exponentially shrinking sheet. Very recently, Najib *et al.* (2014) investigated the effect of surface mass flux on the stagnation point flow over a permeable exponentially stretching/shrinking cylinder.

All the flow field studies mentioned above considered the no-slip boundary condition. However, there are many situations where such condition is invalid and slip may occur on the boundary for particulate fluids, such as foams, emulsions, suspensions and polymer solutions. It is stated in the paper by Cao and Baker (2009) that as the mean free path of the flow becomes comparable to the characteristic length scale of the problem, the flows will start to exhibit non-continuum phenomena as a result of fewer molecular collisions within the dimension of interest. The deviation from interfacial thermodynamic equilibrium will lead to a flow regime where the conventional no-slip wall condition is not valid and the conventional no-slip wall boundary condition fails to accurately model the surface interaction between the fluid and the wall boundary due to the low collision frequency. Therefore, slip models have been proposed to ameliorate the prediction of the non-continuum phenomenon near wall boundaries within the framework of the continuum assumption. The slip boundary condition was proposed by Maxwell (1879). An extensive discussion regarding the slip boundary condition was written by Beavers and Joseph (1967). Quite a number of papers investigating flow field with Navier slip boundary condition are found in the literature, such as Wang (2002), Wang (2003), Wang (2006), Miklavcic and Wang (2004), Ariel (2007), Sajid (2009), Bhattacharyya *et al.* (2011), Aman *et al.* (2013) and Sharma *et al.* (2014). The effects of slip condition can be found easily in open literature, such as Fang *et al.* (2010) and Merkin *et al.* (2012), among others.

Under the Navier slip boundary condition, the slip length is treated as a constant, but according to Thompson and Troian (1997), on the basis of molecular dynamic simulation, the slip length should be a function of shear rate, and they concluded that the slip length behaviour is consistent with Navier slip length at low shear rates. Thompson and Troian (1997) indicated that there exists a general nonlinear relationship between the amount of slip and the local shear rate, and the boundary condition is nonlinear even though it is a Newtonian fluid. Later, Matthews and Hill (2007) discussed the generalized nonlinear Navier boundary condition proposed in (Thompson and Troian 1997) for three flows; through a pipe, a channel and an annulus. Recently, Sajid *et al.* (2010) studied the flows induced by planar and axisymmetric stretching sheet with general boundary condition, while Sajid *et al.* (2012) investigated the axisymmetric stagnation point flow of a viscous fluid over a lubricated surface with a generalized slip boundary condition. The present study extends the idea of Bhattacharyya (2011) by incorporating a general slip boundary condition proposed by Thompson and Troian (1997).

2. Basic Equations

Consider the steady boundary layer flow of a viscous and incompressible fluid past a permeable stretching/shrinking sheet with generalized slip velocity as it is shown in Fig. 1, where x and y are the Cartesian coordinates measured along the sheet and normal to it, respectively, the sheet being located at y = 0. It is assumed that the sheet is stretched/shrinked with the velocity $u_w(x) = U_0 e^{x/L}$, where L is a characteristic length of the sheet, U_0 is the constant velocity characteristic of the sheet. It is also assumed that the temperature of the sheet is $T_w(x) = T_\infty + T_0 e^{x/2L}$, where T_∞ is the ambient

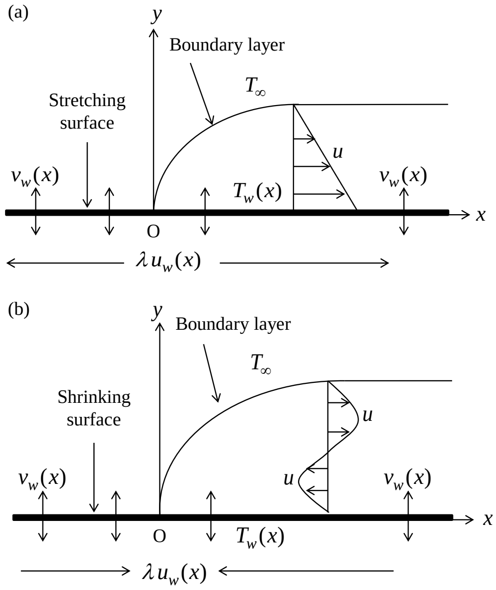

Fig. 1. The geometry of the problem; (a) stretching surface and (b) shrinking surface.

temperature and T_0 is a constant which measures the rate of temperature increase along the sheet.

We also consider that the mass flux velocity is $v_w(x) = v_0 e^{x/2L}$, where v_0 is the constant mass flux velocity with $v_0 < 0$ for suction and $v_0 > 0$ for injection or withdrawal of the fluid, respectively. Under these conditions, the basic boundary layer equations can be written in Cartesian coordinates x and y as (see Bhattacharyya (2011))

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}. \tag{3}$$

Following Thompson and Troian (1997), we assume that the generalized slip velocity condition is given by

$$u_t(x) = \alpha * (1 - \beta * \tau_w)^{-1/2}\tau_w, \tag{4}$$

where u_t is the tangential sheet velocity, $\alpha*$ corresponds to Navier's constant slip length, $\beta*$ is the reciprocal of some critical shear rate and τ_w is the shear stress at the surface of the sheet. Thus, we assume that the boundary conditions of Eqs. (1) to (3) are

$$v_w(x) = v_0 e^{x/2L}, u = \lambda U_0 e^{x/L} + \alpha*(x)\left(1-\beta*(x)\frac{\partial u}{\partial y}\right)^{-\frac{1}{2}}\frac{\partial u}{\partial y},$$
$$T_w(x) = T_\infty + T_0 e^{x/2L} \quad \text{at } y=0, \tag{5}$$
$$u \to 0, T \to T_\infty \quad \text{at } y \to \infty,$$

where u and v are the velocity components along x and y axes, T is the fluid temperature, ν is the kinematic viscosity, ρ is the fluid density, k is the fluid thermal conductivity, c_p is the specific heat at constant pressure and λ is the constant stretching/shrinking parameter with $\lambda > 0$ corresponding to the stretching sheet and $\lambda < 0$ corresponding to the shrinking sheet, respectively .

3. Solution

In order to solve Eqs. (1) to (3) along with the boundary conditions (5), we introduce the following variables:

$$\psi = (2U_0\nu L)^{1/2} e^{x/2L}, \quad \theta(\eta) = \frac{T-T_\infty}{T_w - T_\infty},$$
$$\eta = y\left(\frac{U_0}{2\nu L}\right)^{1/2} e^{x/2L}, \tag{6}$$

where ψ is the stream function with $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. Thus, we have

$$u = u_w(x) f'(\eta),$$
$$v = -\left(\frac{U_0\nu}{2L}\right)^{1/2} e^{x/L}(f(\eta)+\eta f'(\eta)). \tag{7}$$

Thus, we take

$$v_w(x) = -\left(\frac{U_0\nu}{2L}\right)^{1/2} e^{x/L} s, \tag{8}$$

where $s = -v_0/(U_0\nu/2L)^{1/2}$ is the mass flux parameter with $s > 0$ for suction and $s < 0$ for injection or withdrawal of the fluid. Eq. (1) is automatically satisfied, while substituting (6) into Eqs. (2) and (3) yield the following ordinary (similarity) equations:

$$f''' + ff'' - 2f'^2 = 0, \tag{9}$$
$$\frac{1}{Pr}\theta'' + f\theta' - f'\theta = 0, \tag{10}$$

subject to the boundary conditions

$$f(0) = s, f'(0) = \lambda + \frac{\alpha(x)}{\sqrt{(1-\beta(x)f''(0))}} f''(0),$$
$$\theta(0) = 1, \tag{11}$$
$$f'(\eta) \to 0, \theta(\eta) \to 0 \quad \text{as } \eta \to \infty,$$

where primes denote differentiation with respect to η. Further, the three parameters appearing in Eq. (10) and boundary conditions (11) are Pr, $\alpha(x)$ and $\beta(x)$, and they denote the Prandtl number, the velocity slip parameter and the critical shear rate, respectively, which are defined as

$$Pr = \frac{\mu c_p}{k}, \quad \alpha(x) = \sqrt{\frac{a}{2\nu L}} e^{x/2L}\alpha*(x),$$
$$\beta(x) = a\sqrt{\frac{a}{2\nu L}} e^{3x/2L}\beta*(x). \tag{12}$$

As suggested by Aziz (2009), for Eqs. (9) and (10) to have similarity solutions, the quantities $\alpha(x)$ and $\beta(x)$ must be constants and not functions of the variable x as in (12). This condition can be met if $\alpha*(x)$ and $\beta*(x)$ are proportional to $e^{-x/2L}$ and $e^{-3x/2L}$. We therefore assume

$$\alpha*(x) = Ae^{-x/2L}, \quad \beta*(x) = Be^{-3x/2L}, \tag{13}$$

where A and B are constants. With the introduction of (13) into (12), we have

$$\alpha = \sqrt{\frac{a}{2\nu L}} A, \quad \beta = a\sqrt{\frac{a}{2\nu L}} B. \tag{14}$$

Thus, the boundary conditions (11) become

$$f(0) = s, f'(0) = \lambda + \frac{\alpha}{\sqrt{(1-\beta f''(0))}} f''(0),$$
$$\theta(0) = 1, \tag{15}$$
$$f'(\eta) \to 0, \theta(\eta) \to 0 \quad \text{as } \eta \to \infty.$$

We mention that with α and β defined by (14), the solutions of Eqs. (9) and (10) yield the similarity solutions. However, with α and β defined by (13), the solutions generated are the local similarity solutions. We notice that for $\alpha = \beta = 0$, the problem (9)-(11) reduces to the boundary value problems in Elbashbeshy (2001) and Bhattacharyya (2011).

The quantities of physical interest in this problem are the skin friction coefficient C_f and the local Nusselt number Nu_x, which are defined as

$$C_f = \frac{\tau_w}{\rho u_w^2(x)}, \quad Nu_x = \frac{Lq_w}{k(T_w - T_\infty)}, \tag{16}$$

where τ_w and q_w are the skin friction or shear stress along the surface of the sheet and the heat flux from the surface of the sheet, respectively, and are given by

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}, \quad q_w = -k\left(\frac{\partial T}{\partial y}\right)_{y=0}. \tag{17}$$

Using (6), (16) and (17), we get

$$(2Re_x)^{1/2}C_f = f''(0), (2/Re_x)^{1/2} = -\theta'(0), \quad (18)$$

where $Re_x = u_w(x)L/\nu$ is the local Reynolds number.

4. FLOW STABILITY

Weidman *et al.* (2006) and Rosca and Pop (2013) have shown that for the forced convection boundary layer flow past a permeable flat plate and, respectively, for the mixed convection flow past a vertical flat plate, that the lower branch solutions are unstable (not physically realizable), while the upper branch solutions are stable (physically realizable). We test these features by considering the unsteady equations (9) and (10). Following Weidman *et al.* (2006), we introduce the new dimensionless time variable $\tau = t(a/2L)e^{x/L}$. The use of τ is associated with an initial value problem and is consistent with the question of which solution will be obtained in practice (physically realizable). Using the variable τ and (6), we have

$$\begin{aligned} u &= a\frac{\partial}{\partial\eta}f(\eta,\tau), \quad \eta = y\sqrt{\frac{a}{2\nu L}}e^{x/2L}, \\ v &= -\sqrt{\frac{a\nu}{2L}}\left(f(\eta,\tau)+\eta\frac{\partial}{\partial\eta}f(\eta,\tau)\right), \quad (19) \\ \theta(\eta,\tau) &= \frac{T-T_\infty}{T_w-T_\infty}, \quad \tau = t\frac{a}{2L}e^{x/L}, \end{aligned}$$

so that Eqs. (9) and (10) can be written as

$$\frac{\partial^3 f}{\partial\eta^3}+f\frac{\partial^2 f}{\partial\eta^2}-2\left(\frac{\partial f}{\partial\eta}\right)^2-\frac{\partial^2 f}{\partial\eta\partial\tau}=0, \quad (20)$$

$$\frac{1}{Pr}\frac{\partial^2\theta}{\partial\eta^2}+f\frac{\partial\theta}{\partial\eta}-\frac{\partial f}{\partial\eta}\theta-\frac{\partial\theta}{\partial\tau}=0, \quad (21)$$

subject to the boundary conditions

$$\begin{aligned} &f(0,\tau)=s, \theta(0,\tau)=1, \\ &\frac{\partial f}{\partial\eta}(0,\tau)=\lambda+\frac{\alpha}{\sqrt{1-\beta\frac{\partial^2 f}{\partial\eta^2}(0,\tau)}}\frac{\partial^2 f}{\partial\eta^2}(0,\tau) \quad (22) \\ &\frac{\partial f}{\partial\eta}(\eta,\tau)\to 0, \theta(\eta,\tau)\to 0 \quad \text{as } \eta\to\infty. \end{aligned}$$

To test the stability of the steady flow solution $f(\eta)=f_0(\eta)$ and $\theta(\eta)=\theta_0(\eta)$ satisfying the boundary-value problem (20)-(22), we write (see Weidman *et al.* (2006) and Rosca and Pop (2013)),

$$\begin{aligned} f(\eta,\tau) &= f_0(\eta)+e^{-\gamma\tau}F(\eta,\tau), \\ \theta(\eta,\tau) &= \theta_0(\eta)+e^{-\gamma\tau}G(\eta,\tau), \quad (23) \end{aligned}$$

where γ is an unknown eigenvalue parameter, and $F(\eta,\tau)$ and $G(\eta,\tau)$ are small relative to $f_0(\eta)$ and $\theta_0(\eta)$. Substituting (23) into Eqs. (20) and (21), we obtain the following linearized problem:

$$\begin{aligned} &\frac{\partial^3 F}{\partial\eta^3}+f_0\frac{\partial^2 F}{\partial\eta^2}-(4f_0'-\gamma)\frac{\partial F}{\partial\eta} \\ &\quad +f_0''F-\frac{\partial^2 F}{\partial\eta\partial\tau}=0, \quad (24) \end{aligned}$$

$$\begin{aligned} &\frac{1}{Pr}\frac{\partial^2 G}{\partial\eta^2}+f_0\frac{\partial G}{\partial\eta}+F\theta_0'-\theta_0\frac{\partial F}{\partial\eta} \\ &\quad -(f_0'-\gamma)G-\frac{\partial G}{\partial\tau}=0, \quad (25) \end{aligned}$$

along with the boundary conditions

$$\begin{aligned} &F(0,\tau)=0, \frac{\partial G}{\partial\eta}(0,\tau)=0, \\ &\frac{\partial F}{\partial\eta}(0,\tau)=\frac{a}{\sqrt{1-\beta\frac{\partial^2 F}{\partial\eta^2}(0,\tau)}}\frac{\partial^2 F}{\partial\eta^2}(0,\tau), \quad (26) \\ &\frac{\partial F}{\partial\eta}(\eta,\tau)\to 0, G(\eta,\tau)\to 0 \quad \text{as } \eta\to 0. \end{aligned}$$

As suggested in (Weidman *et al.* 2006), we investigate the stability of the steady flow and heat transfer solution $f_0(\eta)$ and $\theta_0(\eta)$ by setting $\tau=0$. Hence, $F=F_0(\eta)$ and $G=G_0(\eta)$ in (24) and (25) identify initial growth or decay of the solution (23). To test our numerical procedure, we have to solve the linear eigenvalue problem

$$F_0'''+f_0F_0''-\left(4f_0'-\gamma\right)F_0'+f_0''F_0=0, \quad (27)$$

$$\frac{1}{Pr}G_0''+f_0G_0'-\left(f_0'-\gamma\right)G_0+F_0\theta_0'-\theta_0F_0' \quad (28)$$

along with the boundary conditions

$$\begin{aligned} &F_0(0)=0, G_0(0)=0, \\ &F_0'(0)=\alpha\left(1-\beta F_0''(0)\right)^{-1/2}, \quad (29) \\ &F_0'(\eta)\to 0, G_0(\eta)\to 0 \quad \text{at } \eta\to\infty. \end{aligned}$$

It should be mentioned that for particular values of γ, Pr, α and β, the stability of the corresponding steady flow solution $f_0(\eta)$ and $\theta_0(\eta)$ are determined by the smallest eigenvalue γ. According to Harris *et al.* (2009), by relaxing a boundary condition on $F_0(\eta)$ or $\theta_0(\eta)$, we can determine the range of possible eigenvalues. For the present problem, we relax the condition that $F_0'\to 0$ as $\eta\to\infty$ and for a fixed value of γ, we solve the system of equations (27) and (28) along with the new boundary condition $F_0''(0)=1$.

5. RESULTS AND DISCUSSION

The nonlinear ordinary differential equations (9) and (10) along with the boundary conditions (11) were solved numerically using the bvp4c function from MATLAB for some values of the governing parameters, namely; suction parameter s, stretching/shrinking parameter λ, velocity slip parameter α, critical shear rate β and Prandtl number Pr. This function is a finite difference code that implements the three-stage Lobatto IIIa formula (see Kierzenka and Shampine (2001) and Shampine *et al.* (2003)). Since the present problem may have multiple (dual) solutions, the bvp4c function requires an initial guess of the solution for (9) and (10). The guess must satisfy the boundary conditions (11) and keep the behaviour of the solution. Determining an initial guess for the upper branch solution is not difficult because the bvp4c method will converge to the first solution even for poor guesses. However, it is rather difficult to determine a sufficiently good guess for the lower branch solution of (9) and (10). In this case, we used the technique called continuation (Shampine *et al.* 2003). The size of the boundary layer thickness is chosen between 4 to 8. To verify the accuracy of the results obtained in this study, the numerical values of the reduced skin friction coefficient $f''(0)$ and the reduced local Nusselt number $-\theta'(0)$ when $\lambda = 1$, $\alpha = \beta = 0$ and $Pr = 0.72$ are compared with those of Elbashbeshy (2001). The comparisons, which are shown in Table 1, are found to be in excellent agreement, and thus we are confident that the present method is accurate.

The variation of the reduced skin friction coefficient $f''(0)$ and the reduced local Nusselt number $-\theta'(0)$ for the no slip case ($\alpha = \beta = 0$) for some values of s and λ are shown in Figs. 2 to 5. Meanwhile, Figs. 6 to 11 display the variation of $f''(0)$ and $-\theta'(0)$ for normal Navier slip ($\alpha \neq 0, \beta = 0$) and generalized slip ($\alpha \neq 0, \beta \neq 0$). It is shown that dual solutions exist for a certain range of suction parameter s for both stretching ($\lambda > 0$) and shrinking ($\lambda < 0$) cases. The first (upper branch) solution and second (lower branch) solutions are illustrated with solid and dashed lines, respectively. It seems that there is no solution for $s < s_c$ and $\lambda < \lambda_c$, where s_c and λ_c are the critical values of s and λ, respectively, beyond which the boundary layer separates from the surface and the solution based upon the boundary layer approximations are not possible.

From these figures, together with the numerical results shown in Table 2, we found that the val-

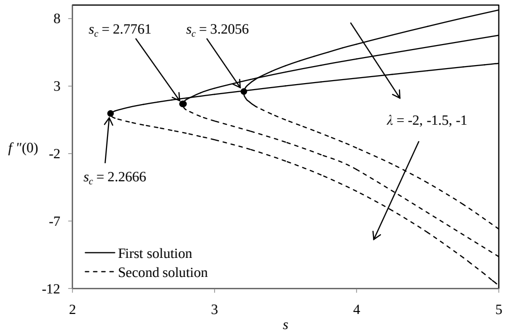

Fig. 2. Variation of $f''(0)$ with s for different λ when $Pr = 0.7$, $\alpha = \beta = 0$ (no slip).

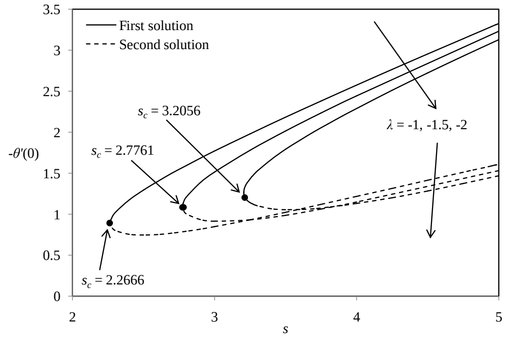

Fig. 3. Variation of $-\theta'(0)$ with s for different λ when $Pr = 0.7$, $\alpha = \beta = 0$ (no slip).

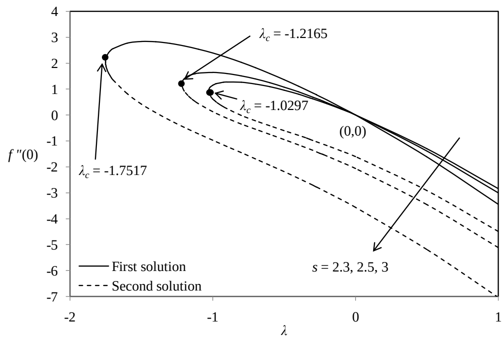

Fig. 4. Variation of $f''(0)$ with λ for different s when $Pr = 0.7$, $\alpha = \beta = 0$ (no slip).

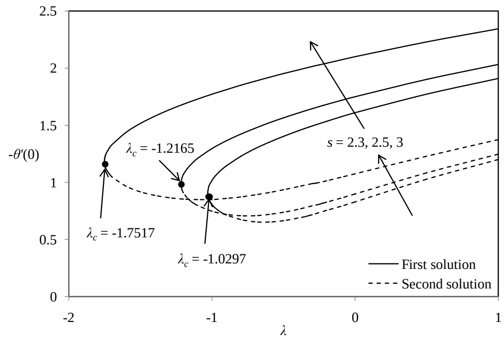

Fig. 5. Variation of $-\theta'(0)$ with λ for different s when $Pr = 0.7$, $\alpha = \beta = 0$ (no slip).

Table 1 Comparison of the values of $-f''(0)$ and $-\theta'(0)$ with those of Elbashbeshy (2001) for different s when $\lambda = 1$ (stretching case), $\alpha = \beta = 0$ (no slip) and $Pr = 0.72$

	Elbashbeshy (2001)		Present study	
s	$-f''(0)$	$-\theta'(0)$	$f''(0)$	$-\theta'(0)$
0	1.28181	0.767778	1.28182	0.767669
0.6	1.59824	1.014517	1.59824	1.014570

Table 2 Values of s_c for several values of α and β when $\lambda = -1$, $Pr = 0.7$

α	β	s_c
0	0	2.2666
1	0	1.6856
	0.5	1.6433
	1	1.5948
5	0	1.1406
	1	1.1157
	2.5	1.0779
	4	1.0421
10	0	0.9321
	1	0.9201
	3	0.8959
	5	0.8735
	7	0.8496

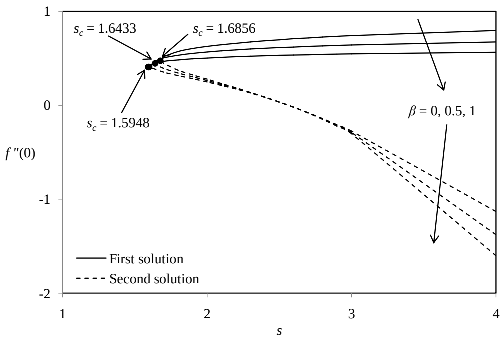

Fig. 6. Variation of $f''(0)$ with s for different β when $\alpha = 1$, $Pr = 0.7$, $\lambda = -1$.

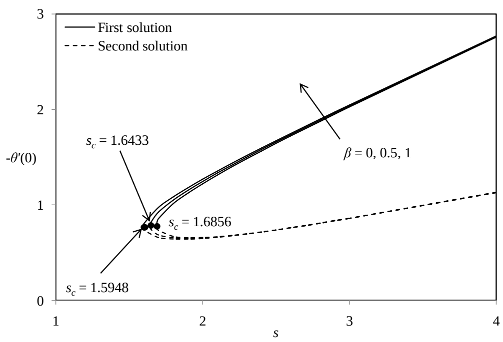

Fig. 7. Variation of $-\theta'(0)$ with s for different β when $\alpha = 1$, $Pr = 0.7$, $\lambda = -1$.

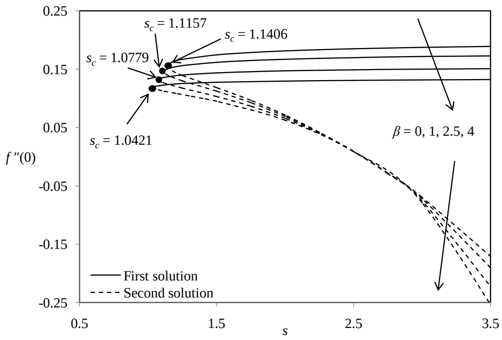

Fig. 8. Variation of $f''(0)$ with s for different β when $\alpha = 5$, $Pr = 0.7$, $\lambda = -1$.

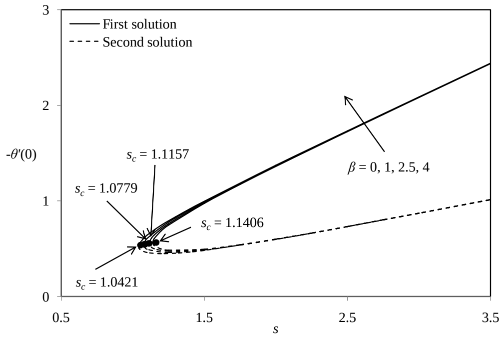

Fig. 9. Variation of $-\theta'(0)$ with s for different β when $\alpha = 5$, $Pr = 0.7$, $\lambda = -1$.

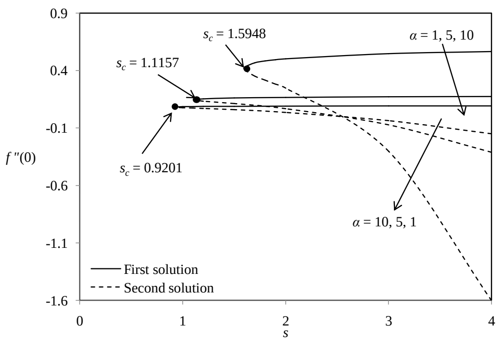

Fig. 10. Variation of $f''(0)$ with s for different α when $\beta = 1$, $Pr = 0.7$, $\lambda = -1$.

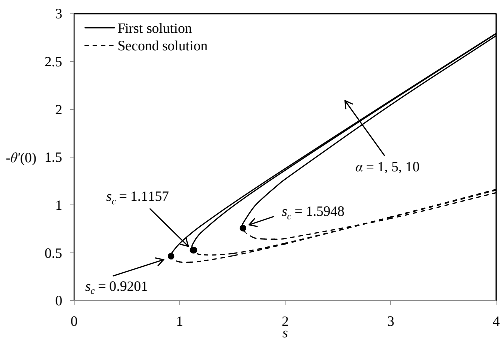

Fig. 11. Variation of $-\theta'(0)$ with s for different α when $\beta = 1$, $Pr = 0.7$, $\lambda = -1$.

ues of s_c increase with the increase of α and β. Hence, the velocity slip parameter α and critical shear rate β widen the range of suction parameter s for which solutions exist. Table 3 shows the numerical results (for both upper and lower

Table 3 Values of $f''(0)$ and $-\theta'(0)$ for several values of λ when $\alpha = 1$, $\beta = 1$, $Pr = 0.7$ and $s = 3$

	First solution		Second solution	
λ	$f''(0)$	$-\theta'(0)$	$f''(0)$	$-\theta'(0)$
1	-0.7571	2.1646	-1.6967	0.8863
0	0.0000	2.1000	-0.9919	0.8487
-1	0.7413	2.0263	-0.2751	0.8605
-2	1.4601	1.9388	0.4829	0.9235
-3	2.1419	1.8271	1.3083	1.0389
-4	2.7345	1.6511	2.2551	1.2347
-4.3774 (=λ_c)	2.7915	1.4498	2.7868	1.4458

Table 4 Smallest eigenvalues of γ for several values of α, β and s when $\lambda = -1$ and $Pr = 0.7$

α	β	s	γ (upper branch)	γ (lower branch)
0	0	2.3	0.3312	-0.3275
		2.4	0.6708	-0.6588
		2.5	0.8979	-0.8811
		3	1.0881	-1.0682
1	0	1.7	0.1658	-0.1660
		1.8	0.4724	-0.4763
		1.9	0.6544	-0.6646
		2	0.8049	-0.8229
	0.5	1.7	0.1022	-0.1217
		1.8	0.3168	-0.3380
		1.9	0.5379	-0.5652
		2	0.7000	-0.7358

branches) of $f''(0)$ and $-\theta'(0)$ for several values of stretching/shrinking parameter λ when $\alpha = \beta = 1$, $Pr = 0.7$ and $s = 3$. It can be seen that the values of $f''(0)$ increase while the values of $-\theta'(0)$ decrease with the increase of $|\lambda_c|$, and the solutions exist up to a critical value of λ, which in this case, is $\lambda_c = -4.3774$.

Figures 12 and 13 display the velocity profiles $f'(\eta)$ and temperature profiles $\theta(\eta)$, respectively, for different values of s when $\alpha = 1$, $\beta = 0.5$, $\lambda = 1$ and $Pr = 0.7$. Both figures show

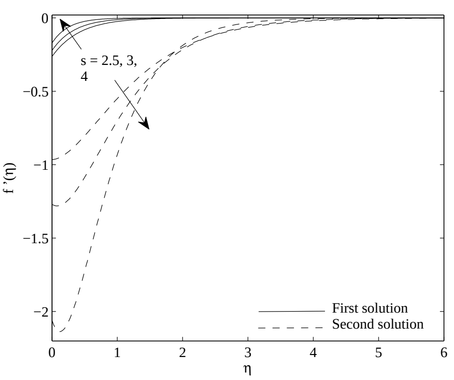

Fig. 12. Velocity profiles $f'(\eta)$ for different values of s when $\alpha = 1$, $\beta = 0.5$, $\lambda = -1$, $Pr = 0.7$.

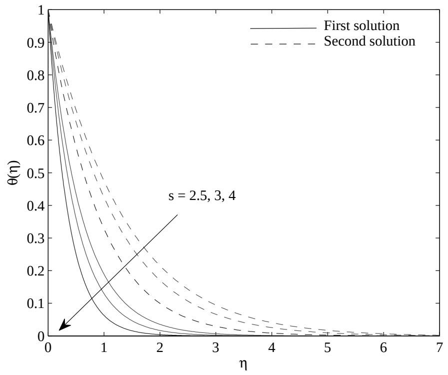

Fig. 13. Temperature profiles $\theta(\eta)$ for different values of s when $\alpha = 1$, $\beta = 0.5$, $\lambda = -1$, $Pr = 0.7$.

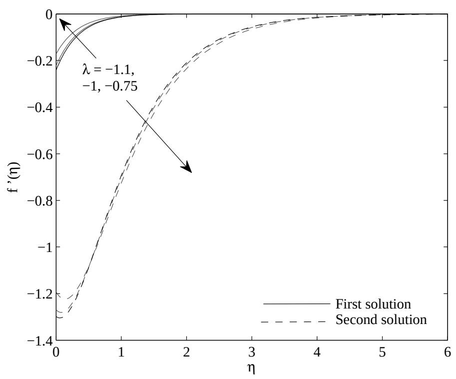

Fig. 14. Velocity profiles $f'(\eta)$ for different values of λ when $\alpha = 1$, $\beta = 0.5$, $s = 3$, $Pr = 0.7$.

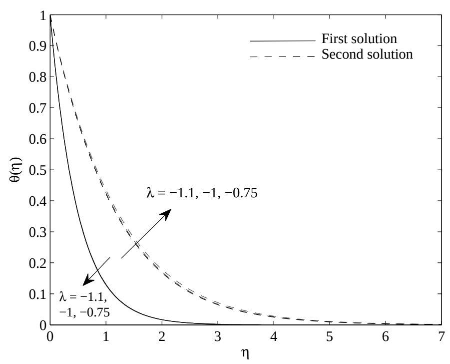

Fig. 15. Temperature profiles $\theta(\eta)$ for different values of λ when $\alpha = 1$, $\beta = 0.5$, $s = 3$, $Pr = 0.7$.

the reduction in boundary layer thickness with the increase of suction parameter s. This happened due to the reduced drag force cause by suction ($s > 0$) in order to avoid boundary layer separation. Meanwhile, Figs. 14 and 15 illustrate the velocity profiles $f'(\eta)$ and temperature profiles $\theta(\eta)$, respectively, for different values of λ when $\alpha = 1$, $\beta = 0.5$, $s = 3$ and $Pr = 0.7$.

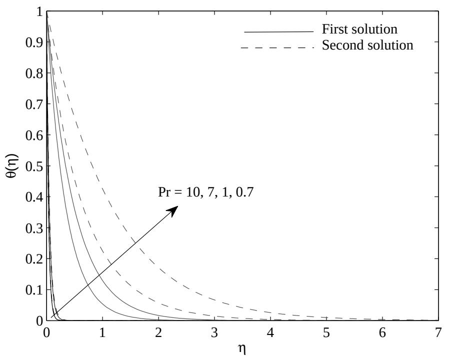

Fig. 16. Temperature profiles $\theta(\eta)$ for different values of *Pr* when $\alpha = 1$, $\beta = 0.5$, $s = 3$, $\lambda = -1$

It can be seen that the boundary layer thickness decreases with the increase of λ. Since Eqs. (9) and (10) are uncoupled, the changes in Prandtl number *Pr* has no influence on the flow field. Fig. 16 shows the effect of the Prandtl number *Pr* to the temperature profiles $\theta(\eta)$ when $\alpha = 1$, $\beta = 0.5$, $s = 3$ and $\lambda = -1$. The boundary layer thickness is shown to be smaller with larger number of *Pr*. The boundary layer thickness for the second (lower branch) solution is always larger than the first (upper branch) solution, as can be observed from Figs. 12 to 16. It is worth mentioning that the computation was made until the solution exists up to the smallest value of s and λ where both velocity $f'(\eta)$ and temperature $\theta(\eta)$ profiles satisfy the far field boundary conditions (11) asymptotically, hence supporting the numerical results obtained.

The dual solutions are very important because quite different flow behaviour is observed for a shrinking sheet than for a stretching sheet. This new type of shrinking sheet flow is essentially a backward flow as discussed by Goldstein (2006). In addition, it should be mentioned that it is evident in the paper by Weidman *et al.* (2006) that there exist critical values of the suction parameter s, and it applies also in the case of the present problem as can be seen in Figs. 6 to 9. Between the two solutions obtained in this study, we expect that the first (upper branch) solution is stable and physically relevant with real world applications while the second (upper branch) solution is not. A stability analysis was performed by solving an unknown eigenvalue γ on Eqs. (27)-(28), along with the boundary conditions (29) to determine which of the solution is stable. The smallest eigenvalues γ for different values of α, β and s are shown in Table 4. From the table, it is seen that the upper branch solutions have positive eigenvalues γ while the lower branch solutions have negative eigenvalues γ, thus we conclude that the first (upper branch) solution is stable while the second (lower branch) solution is unstable.

6. Conclusion

A numerical study was performed for the problem of boundary layer flow and heat transfer over a permeable exponentially stretching/shrinking sheet with generalized slip velocity. The problem was solved by using "bvp4c" function in MATLAB. The numerical results obtained were compared with the previous literature and the comparison is found to be in good agreement. The boundary layer thickness was found to be smaller with increasing suction parameter, stretching/shrinking parameter and Prandtl number. The boundary layer thickness of the second (lower branch) solution appeared to be larger than the first (upper branch) solution. The introduction of the generalized slip boundary condition resulted in the reduction of the local skin friction coefficient and local Nusselt number. Dual solutions were found for a certain range of the suction and stretching/shrinking parameter. Stability analysis was performed and concluded that the first (upper branch) solution was stable while the second (lower branch) solution was not.

Acknowledgments

This work was supported by research grants (DIP-2015-010 and FRGSTOP-DOWN/2014/SG04/UKM/01/ 1) from the Universiti Kebangsaan Malaysia and the Ministry of Higher Education, Malaysia, respectively. The authors wish to express their thanks to the very competent Reviewers for the valuable comments and suggestions.

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