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Flow of Generalized Burgers' Fluid between Side Walls Induced by Sawtooth Pulses Stress

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ABSTRACT

This paper presents the unsteady magnetohydrodynamic (MHD) flow of a generalized Burgers' fluid between two parallel side walls perpendicular to a plate. The flow is generated from rest at time $t = 0^+$ induced by sawtooth pulses stress applied to the bottom plate. The solutions obtained by means of the Laplace and the Fourier cosine and sine transforms in this order are presented as a sum between the corresponding Newtonian and non-Newtonian contributions. We investigate the effect of magnetic field and permeability on the fluid motion by a numerical procedure for the inverse Laplace transform, namely Stehfest's algorithm. Moreover, the influence of side walls on the fluid motion, the effect of pulse period, magnetic and porosity parameters and material parameters is presented by graphical illustrations.

Keywords: Generalized burgers' fluid; Sawtooth pulses stress; MHD flow; Porous medium.

NOMENCLATURE

A	first Rivlin-Ericksen tensor	V	velocity field
B	total magnetic field		
$H(\cdot)$	Heaviside unit step function	σ	electrical conductivity of the fluid
J	current density	a_1	relaxation time
$\hat{k}$	unit vector along the z-direction	a_3	retardation time
k	permeability of the medium	a_2	material parameter
L	velocity gradient	a_4	material parameter
R	Darcy's resistance	¬	kinematic viscosity
S	extra-stress tensor	${}_0$	strength of applied magnetic field
T	time period		

1. INTRODUCTION

The interest in the research of oscillating motions of non-Newtonian fluids induced by stress has been greatly grown due to their multiple applications. Such type of flows are of interest to workers in cement industry, medicine and geology, e.g., in dams, clay rotation, artificial surfing, heartbeat, motion in the liquid core of the earth during earthquakes are the best examples of motion induced by stress. Among the many models that have been used to describe the behavior of rate type fluids, Burgers' fluids are the most complex one. There is a long chain of work published on rate type models, but we mention here the references that most relate to our work.

Chakraborty and Ray (1980) examined the magneto-hydrodynamic Couette flow between two parallel plates when one of the plates was set into motion by random pulses applied at random instants of time. Makar (1987) presented the solution of magneto-hydrodynamic flow between two parallel plates when one of the plates was subjected to velocity tooth pulses, neglecting the induced magnetic field. Ghosh and Sana (2009) solved an initial value problem designed for the motion of an Oldroyd-B fluid bounded by an infinite rigid non-conducting plate. The flow was generated from rest in the fluid due to velocity sawtooth pulses subjected on the plate in the presence of an external magnetic field. Khan and Zeeshan (2011) investigated the MHD flow of an Oldroyd-B fluid through a porous space. The motion was generated in the fluid due to the velocity sawtooth pulses applied on the plate.

In all of the above citations, the conditions on the boundary are given in terms of velocity. A little work is available in the literature where stress is given on the boundary. It is therefore of interest to construct solution of hydromagnetic flow of fluids induced by stress. The stress at the boundary gives important information about the nature of dissipation at the boundary. Fetecau *et al*. (2011) obtained solutions for the motion of viscous fluids due to an infinite plate that applies oscillating shear stresses to the fluid. Seth *et al*. (2011) studied the unsteady hydromagnetic couette flow of an incompressible electrically conducting fluid in a rotating system in the presence of a uniform transverse magnetic field through porous medium induced by the impulsive movement of the upper plate of the channel using Laplace transform technique. Vieru (2011) obtained the solution for the flow induced by a flat plate between two walls perpendicular to a plate that applied a time dependent shear to the fluid. Vieru *et al*. (2012) examined the unsteady motion of a second grade fluid between two parallel side walls perpendicular to a plate by means of the Fourier sine and cosine transforms. The flow was induced by an infinite plate that applied an oscillating shear to the fluid.

Fetecau *et al*. (2012) presented an analysis for the unsteady flow of an incompressible Oldroyd-B fluid induced by a time-dependent shear stress of order of fractional parameter to the fluid using Fourier sine and Laplace transforms. Sohail *et al*. (2013) provided close-form expressions for the starting solutions corresponding to the unsteady motion of a Maxwell fluid between two parallel walls due to an infinite plate that applies oscillating shear stresses to the fluid. Rubbab *et al*. (2013) studied the unsteady natural convection flow of an incompressible viscous fluid near a vertical plate that applied an arbitrary shear stress to the fluid using the Laplace transform technique.

Seth and Singh (2013) presented an analysis for the unsteady hydromagnetic couette flow of a viscous incompressible electrically conducting fluid in a rotating system with Hall effects in the presence of a uniform transverse magnetic field induced by the time dependent velocity oscillations of the upper plate in its own plane. Kumar and Prasad (2014) presented the MHD pulsatile flow through a porous medium driven by an unsteady pressure gradient between permeable beds of a viscous incompressible Newtonian fluid. Further studies of fluid motions induced by stress may be found in Akhtar *et al*. (2011), Shahid *et al*. (2012), Sultan *et al*. (2014) and Sultan *et al*. (2015).

The main objective of the present investigation is to study the MHD flow of a generalized Burgers' fluid through a porous medium between two parallel walls. The flow is induced by the sawtooth pulses stress. The MHD flow involving such fluids has promising applications on the development of energy generation MHD pumps. The boundary condition used is of interest as sound waves, light waves, and ocean waves travel like the form of sawtooth pulses. Study of sawtooth pulses flows is also important because of their increasing applications in aerospace science, astrophysics, atmospheric science and physiological fluid dynamics Ghosh (2011). Moreover, a sawtooth wave has strong electromagnetic properties. A sawtooth pulse may be used in the detection of magnetic effect on electrically conducting flows of biological fluids. Analytical expressions for the velocity field and the shear stresses are determined by means of the Fourier cosine and sine transforms coupled with Laplace transform. Finally, a comprehensive study of some physical parameters involved is performed to illustrate the influence of these parameters on the velocity.

2. GOVERNING EQUATIONS

For the generalized Burgers' fluids, the Cauchy stress tensor is given by

$$\tau = -p\,I + S, \tag{1}$$

$$S + \lambda_1 \frac{\delta S}{\delta t} + \lambda_2 \frac{\delta^2 S}{\delta t^2} = \mu\left(A + \lambda_3 \frac{\delta A}{\delta t} + \lambda_4 \frac{\delta^2 A}{\delta t^2}\right), \tag{2}$$

where $-pI$ denotes the indeterminate spherical stress, $\lambda_3 < \lambda_1$ having the dimension of time and λ_2 and λ_4 both having the dimension of square of time and

$$\frac{\delta^2 S}{\delta t^2} = \frac{\delta}{\delta t}\left(\frac{ds}{dt} - LS - SL^T\right). \tag{3}$$

We seek a velocity field V and stress field S of the form Sultan *et al*. (2014)

$$V = v(x, y, t) = w(x, y, t)\,\hat{k}, \quad S = S(x, y, t), \tag{4}$$

where $\hat{k}$ is the unit vector along the z -direction. If the fluid is at rest up to the moment $t = 0$, then

$$V(x, y, 0) = 0, \quad S(x, y, 0) = \frac{\partial S(x, y, 0)}{\partial t} = 0. \tag{5}$$

Eqs. (1)- (3) and (5) give $S_{xx} = S_{xy} = S_{yy} = 0$ and the meaningful equations

$$\left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right)\tau_1(x, y, t) = \mu\left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right)\frac{\partial w(x, y, t)}{\partial x} \quad \text{for } \tau_1(x, y, 0) = 0. \tag{6}$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right)\tau_2(x, y, t) = \mu\left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right)\frac{\partial w(x, y, t)}{\partial y} \quad \text{for } \tau_2(x, y, 0) = 0. \tag{7}$$

where $\tau_1 = S_{xz}(x, y, t)$ and $\tau_2 = S_{yz}(x, y, t)$ are the non-trivial shear stresses.

The Darcy's resistance R in a generalized Burgers' fluid satisfies the following expression

$$(1+ª_1\frac{\partial}{\partial t}+ª_2\frac{\partial^2}{\partial t^2})R=$$

$$-\frac{«¤}{k}(1+ª_3\frac{\partial}{\partial t}+ª_4\frac{\partial^2}{\partial t^2})V, \qquad (8)$$

We assume that a uniform Magnetic field of strength, $_0$, is applied to the fluid. We also assume that the direction of magnetic field is perpendicular to the velocity field.

Thus the Lorentz force due to magnetic field becomes

$$J\times B=-± \ _0^2 V, \qquad (9)$$

The balance of linear momentum which governs the MHD flow through porous medium becomes

$$° \frac{dV}{dt}=\nabla\cdot{}^2 + J\times B + R, \qquad (10)$$

We consider the unsteady flow of an incompressible generalized Burgers' fluid over an infinite flat plate between two parallel side walls separated by a distance d apart, perpendicular to the plate. Initially, the system is at rest. At time $t=0^+$, the plate applies a pulsating shear to the fluid induced by sawtooth pulses.

In view of Eqs. (6)-(10), the governing equation leads to

$$\left(1+ª_1\frac{\partial}{\partial t}+ª_2\frac{\partial^2}{\partial t^2}\right)\frac{\partial w(x,y,t)}{\partial t}=$$

$$\left(1+ª_3\frac{\partial}{\partial t}+ª_4\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)w(x,y,t)$$

$$-\Omega\left(1+ª_1\frac{\partial}{\partial t}+ª_2\frac{\partial^2}{\partial t^2}\right)w(x,y,t)$$

$$-£\left(1+ª_3\frac{\partial}{\partial t}+ª_4\frac{\partial^2}{\partial t^2}\right)w(x,y,t). \qquad (11)$$

where $¬=\frac{«}{°}, \Omega=\frac{± \ _0^2}{°}$ and $£=\frac{¬¤}{k}$.

We use the following appropriate initial conditions

$$w(x,y,0)=\frac{\partial w(x,y,0)}{\partial t}=\frac{\partial^2 w(x,y,0)}{\partial t^2}=0,$$

for $x>0,\ y\in[0,d]$, (12)

the boundary conditions

$$(1+ª_1\frac{\partial}{\partial t}+ª_1\frac{\partial^2}{\partial t^2})¿_1(x,y,t)|_{x=0}=$$

$$«(1+ª_3\frac{\partial}{\partial t}+ª_4\frac{\partial^2}{\partial t^2})\frac{\partial w(x,y,t)}{\partial x}|_{x=0}=U\ f(t)$$

for $y\in(0,d)$ and $t>0$, (13)

$$w(x,0,t)=w(x,d,t)=0 \text{ for } x,t>0. \qquad (14)$$

and the natural conditions

$$w(x,y,t)=\frac{\partial w(x,y,t)}{\partial x}\to 0 \text{ as } x\to\infty, y\in[0,d]. \qquad (15)$$

where $f(t)$ denotes the sawtooth pulses which is an even periodic function.

According to the nature of the applied stress, we assume that the mathematical form of the function $f(t)$, Ghosh, A.K. and P. Sana (2009), is

$$f(t)=\frac{1}{T}\left(tH(t)+\frac{1}{2}\sum_{p=1}^{\infty}(-1)^p(t-pT)H_{pT}(t)\right), \qquad (16)$$

Where $H(\cdot)$ is defined as $H_{pT}(t)=0$ for $t\le pT$ and $H_{pT}(t)=1$ for $t>pT$.

In order to solve the problem, we use the Laplace transform technique and Fourier cosine and sine transforms in this order.

3. CALCULATION OF VELOCITY FIELD

Applying the Laplace transform to Eq. (11), we obtain the following problem

$$\left(1+ª_1 q+ª_2 q^2\right)q\overline{w}(x,y,q)=$$

$$¬\left(1+ª_3 q+ª_4 q^2\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\overline{w}(x,y,q)$$

$$-\Omega\left(1+ª_1 q+ª_2 q^2\right)\overline{w}(x,y,q)$$

$$-£\left(1+ª_3 q+ª_4 q^2\right)\overline{w}(x,y,q). \qquad (17)$$

The Laplace transform $\overline{w}(x,y,q)$ of the function $w(x,y,t)$ has to satisfy the conditions

$$(1+ª_1 q+ª_2 q^2)\overline{¿}_1(x,y,q)|_{x=0}$$

$$=«(1+ª_3 q+ª_4 q^2)\frac{\partial\overline{w}(x,y,q)}{\partial x}|_{x=0}$$

$$=\frac{U}{T}\left(\frac{1}{q^2}+\frac{1}{2}\sum_{p=1}^{\infty}(-1)^p\exp(-pTq)\frac{1}{q^2}\right), \qquad (18)$$

$$w(x,0,q)=w(x,d,q)=0. \qquad (19)$$

$$\overline{w}(x,y,q)=\frac{\partial\overline{w}(x,y,q)}{\partial x}\to 0 \text{ as}$$

$$x\to\infty, y\in[0,d]. \qquad (20)$$

Multiplying both sides of Eq. (17) by $\sqrt{\frac{2}{®}}\cos(¸ x)\sin(ª_n y)$, where $ª_n=\frac{n®}{d}$, integrating with respect to x and y from 0 to ∞ and 0 to d respectively, and bearing in mind the conditions (18)-(20), we find that

$$\overline{w}_n(¸,q)=\sqrt{\frac{2}{®}}\frac{U}{T°}\frac{(-1)^n-1}{ª_n}$$
$$\times 1/[(ª_2 q^3+(ª_1+¬ª_4(¸^2+ª^2_n)+ª_2\Omega+ª_4£)q^2$$
$$+(1+¬ª_3(¸^2+ª^2_n)+ª_1\Omega+ª_3£)$$
$$+¬(¸^2+ª^2_n)+\Omega+£)]$$
$$\times\frac{1}{q^2}\left[1+2\sum_{p=1}^{\infty}(-1)^p\exp(-pTq)\right], \quad (21)$$

Where

$$\overline{w}_n(¸,q)=\sqrt{\frac{2}{®}}\int_0^{\infty}\int_0^{d}\overline{w}(x,y,q)\cos(¸x)$$

$$\sin(ª_n y)\,dy\,dx;\ n=1,2,3,\ldots. \quad (22)$$

Eq. (21) can also be written as

$$\overline{w}_n(¸,q)=\sqrt{\frac{2}{®}}\frac{U}{T°}\frac{(-1)^n-1}{ª_n}[\frac{1}{q+q_{4,n}(¸)}$$
$$-[\{ª_2q^3+\left(ª_1+¬ª_4(¸^2+ª^2_n)+ª_2\Omega+ª_4£\right)q^2$$
$$+\left(¬ª_3(¸^2+ª^2_n)+ª_1\Omega+ª_3£\right)q\}$$
$$/\{(ª_2q^3+\left(ª_1+¬ª_4(¸^2+ª^2_n)+ª_2\Omega+ª_4£\right)q^2$$
$$+\left(1+¬ª_3(¸^2+ª^2_n)+ª_1\Omega+ª_3£\right)q+¬(¸^2+ª^2_n)$$
$$+\Omega+£)(q+q_{4,n}(¸))\}]]$$
$$\times\frac{1}{q^2}\left[1+2\sum_{p=1}^{\infty}(-1)^p\exp(-pTq)\right]. \quad (23)$$

or

$$\overline{w}_n(¸,q)=\sqrt{\frac{2}{®}}\frac{U}{T°}\frac{(-1)^n-1}{ª_n}[\frac{1}{q+q_{4,n}(¸)}$$
$$-\frac{1}{ª_2}[\{ª_2q^3+(ª_1+¬ª_4(¸^2+ª^2_n)+ª_2\Omega+ª_4£)q^2$$
$$+(¬ª_3(¸^2+ª^2_n)+ª_1\Omega+ª_3£)q\}$$
$$/\{(q-q_{1,n}(¸))(q-q_{2,n}(¸))(q-q_{3,n}(¸))$$
$$(q+q_{4,n}(¸))\}]]$$

$$\times\frac{1}{q^2}(1+2\sum_{p=1}^{\infty}(-1)^p\exp(-pTq)). \quad (24)$$

Writing the above Eq. as

$$\overline{w}_n(¸,q)=\sqrt{\frac{2}{®}}\frac{U}{T°}\frac{(-1)^n-1}{ª_n}[\frac{1}{q+q_{4,n}(¸)}$$
$$-\frac{1}{ª_2}[\frac{¨_{1,n}(¸)}{q-q_{1,n}(¸)}+\frac{¨_{2,n}(¸)}{q-q_{2,n}(¸)}$$
$$+\frac{¨_{3,n}(¸)}{q-q_{3,n}(¸)}+\frac{¨_{4,n}(¸)}{q+q_{4,n}(¸)}]]$$
$$\times\frac{1}{q^2}(1+2\sum_{p=1}^{\infty}(-1)^p\exp(-pTq)). \quad (25)$$

where

$$¨_{1,n}(¸)=¤_{1,n}(¸)/[(q_{1,n}(¸)-q_{2,n}(¸))$$
$$(q_{1,n}(¸)-q_{3,n}(¸))(q_{1,n}(¸)+q_{4,n}(¸))] \quad (26)$$

$$¨_{2,n}(¸)=¤_{2,n}(¸)/[(q_{2,n}(¸)-q_{1,n}(¸))$$
$$(q_{2,n}(¸)-q_{3,n}(¸))(q_{2,n}(¸)+q_{4,n}(¸))] \quad (27)$$

$$¨_{3,n}(¸)=¤_{3,n}(¸)/[(q_{3,n}(¸)-q_{1,n}(¸))$$
$$(q_{3,n}(¸)-q_{2,n}(¸))(q_{3,n}(¸)+q_{4,n}(¸))] \quad (28)$$

$$¨_{4,n}(¸)=¤_{4,n}(¸)/[(q_{4,n}(¸)-q_{1,n}(¸))$$
$$(q_{4,n}(¸)-q_{2,n}(¸))(q_{4,n}(¸)-q_{3,n}(¸))] \quad (29)$$

$$q_{4,n}(¸)=¬(¸^2+ª^2_n)+\Omega+£ \quad (30)$$

$$q_{i,n}(¸)=s_{i,n}(¸)$$
$$-\frac{ª_1+¬ª_4(¸^2+ª^2_n)+ª_2\Omega+ª_4£}{3ª_2^2} \quad (31)$$

$$¤_{i.n}(¸)=ª_2q^3_{i,n}(¸)+(ª_1+¬ª_4(¸^2+ª^2_n)$$
$$+ª_2\Omega+ª_4£)q^2_{i,n}(¸)+(¬ª_3(¸^2+ª^2_n)$$
$$ª_1\Omega+ª_3£)q_{i,n}(¸)\ \text{for n=1,2,3.} \quad (32)$$

$$¤_{4.n}(¸)=-ª_2q^3_{4,n}(¸)+(ª_1+¬ª_4(¸^2+ª^2_n)$$
$$+ª_2\Omega+ª_4£)q^2_{4,n}(¸)-(¬ª_3(¸^2+ª^2_n)$$
$$ª_1\Omega+ª_3£)q_{4,n}(¸)$$

In the above relations

$$s_{1,n}(¸)=\left(-\frac{_{1,n}(¸)}{2}+\sqrt{\frac{^2_{1,n}(¸)}{4}+\frac{~^3_{1,n}(¸)}{27}}\right)^{1/3}$$
$$+\left(-\frac{_{1,n}(¸)}{2}-\sqrt{\frac{^2_{1,n}(¸)}{4}+\frac{~^3_{1,n}(¸)}{27}}\right)^{1/3} \quad (33)$$

$$s_{2,n}(¸)=Z\left(-\frac{_{1,n}(¸)}{2}+\sqrt{\frac{^2_{1,n}(¸)}{4}+\frac{~^3_{1,n}(¸)}{27}}\right)^{1/3}$$
$$+Z^2\left(-\frac{_{1,n}(¸)}{2}-\sqrt{\frac{^2_{1,n}(¸)}{4}+\frac{~^3_{1,n}(¸)}{27}}\right)^{1/3}, \quad (34)$$

$$s_{3,n}(¸)=Z^2\left(-\frac{_{1,n}(¸)}{2}+\sqrt{\frac{^2_{1,n}(¸)}{4}+\frac{~^3_{1,n}(¸)}{27}}\right)^{1/3}$$
$$+Z\left(-\frac{_{1,n}(¸)}{2}-\sqrt{\frac{^2_{1,n}(¸)}{4}+\frac{~^3_{1,n}(¸)}{27}}\right)^{1/3}, \quad (35)$$

Where

$$\sim_{1,n}(¸)=\frac{1+¬ª_3(¸^2+ª_n^2)+ª_1\Omega+ª_3£}{ª_2}$$

$$-\frac{(ª_1+¬ª_4(¸^2+ª_n^2)+ª_2\Omega+ª_4£)^2}{3ª_2^{\,2}}, \qquad (36)$$

$$_{1,n}(¸)=\frac{¬(¸^2+ª_n^2)+\Omega+£}{ª_2}$$

$$+2\frac{(1+¬ª_3(¸^2+ª_n^2)+ª_1\Omega+ª_3£)^3}{27ª_2^{\,3}}$$

$$-\frac{1}{3ª_2^{\,2}}\{(ª_1+¬ª_4(¸^2+ª_n^2)+ª_2\Omega+ª_4£)$$

$$\times(1+¬ª_3(¸^2+ª_n^2)+ª_1\Omega+ª_3£)\}, \quad (37)$$

$$Z=\frac{-1+i\sqrt{3}}{2}. \qquad (38)$$

To solve Eq. (25), we use the formula

$$L^{-1}\left(\frac{1}{q^2(q-q_{1,n}(¸))}\right)=$$

$$\frac{\exp(q_{1,n}(¸)t)-q_{1,n}(¸)t-1}{q_{1,n}^2(¸)}. \qquad (39)$$

Inversion of Eq. (25) by means of the Laplace transform and Fourier cosine and sine transforms, and using Eq. (39), we obtain

$$w(x,y,t)=\frac{2}{d}\sum_{n=1}^{\infty}\sqrt{\frac{2}{®}}\sin(ª_n y)\int_0^{\infty}B_n(¸)\cos(¶x)$$

$$\times[-\frac{1}{q_{1,n}^2(¸)}\{(\exp(q_{1,n}(¸)t)-q_{1,n}(¸)t-1)H(t)$$

$$+(\exp(q_{1,n}(¸)(t-pT))-q_{1,n}(¸)$$

$$\times(t-pT)-1)H_{pT}(t)\}]d¸$$

$$-\frac{2}{d}\sum_{n=1}^{\infty}\sqrt{\frac{2}{®}}\sin(ª_n y)\int_0^{\infty}A_n(¸)\cos(¶x)$$

$$\times[\{\ell_{1,n}(¸,t)+\ell_{2,n}(¸,t)+\ell_{3,n}(¸,t)\}H(t)$$

$$+2\sum_{p=1}^{\infty}(-1)^p\{\ell_{1,n}(¸,t-pT)+\ell_{2,n}(¸,t-pT)$$

$$+\ell_{3,n}(¸,t-pT)\}H_{pT}(t)]d¸. \qquad (40)$$

Where

$$A_n(¸)=\sqrt{\frac{2}{®}}\frac{U}{T°}\frac{(-1)^n-1}{ª_nª_2},B_n(¸)=\sqrt{\frac{2}{®}}\frac{U}{T°}\frac{(-1)^n-1}{ª_n},$$

$$\ell_{j,n}(¸,t)=¨_{j,n}(¸)\frac{\exp(q_{j,n}(¸)t)-q_{j,n}(¸)t-1}{q_{j,n}^2(¸)}$$

for $j=1,2,3.$ (41)

The first part of Eq. (40) gives solution for Newtonian fluid while the second part gives corresponding non-Newtonian contribution.

4. CALCULATION OF TANGENTIAL STRESS

To obtain the expressions for the shear stresses $²_1(x,y,t)$ and $²_2(x,y,t)$, applying the Laplace transform to Eqs. (6) and (7), we have the expressions

$$\bar{²}_1(x,y,q)=\frac{(1+ª_3q+ª_4q^2)}{(1+ª_1q+ª_2q^2)}\frac{\partial\bar{w}(x,y,q)}{\partial x} \qquad (42)$$

$$\bar{²}_2(x,y,q)=\frac{(1+ª_3q+ª_4q^2)}{(1+ª_1q+ª_2q^2)}\frac{\partial\bar{w}(x,y,q)}{\partial y} \qquad (43)$$

From Eq. (21) with inverse Fourier cosine and sine transforms, we have

$$\bar{w}(x,y,q)=\frac{2}{d}\sum_{n=1}^{\infty}\sqrt{\frac{2}{®}}A_n(¸)\sin(ª_n y)\cos(¸x)$$

$$\times\frac{1}{(q-q_{1,n}(¸))(q-q_{2,n}(¸))(q-q_{3,n}(¸))}$$

$$\times\frac{1}{q^2}(1+2\sum_{p=1}^{\infty}(-1)^p\exp(-pTq))d¸. \qquad (44)$$

Using Eq. (44) in Eqs. (42) and (43) , we have

$$\bar{²}_1(x,y,q)=-\frac{2«}{d}\sum_{n=1}^{\infty}\sqrt{\frac{2}{®}}A_n(¸)\sin(ª_n y)$$

$$\times\int_0^{\infty}\frac{¸\sin(¸x)(1+ª_3q+ª_4q^2)}{(1+ª_1q+ª_2q^2)}$$

$$\times\frac{1}{(q-q_{1,n}(¸))(q-q_{2,n}(¸))(q-q_{3,n}(¸))}$$

$$\times\frac{1}{q^2}(1+2\sum_{p=1}^{\infty}(-1)^p\exp(-pTq))d¸. \qquad (45)$$

$$\bar{²}_2(x,y,q)=\frac{2«}{d}\sum_{n=1}^{\infty}\sqrt{\frac{2}{®}}ª_nA_n(¸)\cos(ª_n y)$$

$$\times\int_0^{\infty}\frac{\cos(¸x)(1+ª_3q+ª_4q^2)}{(1+ª_1q+ª_2q^2)}$$

$$\times\frac{1}{(q-q_{1,n}(¸))(q-q_{2,n}(¸))(q-q_{3,n}(¸))}$$

$$\times\frac{1}{q^2}(1+2\sum_{p=1}^{\infty}(-1)^p\exp(-pTq))d¸. \qquad (46)$$

Let us take

$$\bar{A}(q)=\frac{(1+ª_3q+ª_4q^2)}{(1+ª_1q+ª_2q^2)}, \qquad (47)$$

and

$$\bar{B}_n(¸,q)=$$

$$\frac{1}{q^2(q-q_{1,n}(\xi))(q-q_{2,n}(\xi))(q-q_{3,n}(\xi))}, \quad (48)$$

Writing Eq. (47) under the following equivalent form

$$\overline{A}(q)=a_2+a_3\frac{q+a_1}{(q+a_1)^2-b_1^2}$$

$$+a_4\frac{b_1}{(q+a_1)^2-b_1^2}, \quad (49)$$

Where

$$a_1=\frac{\lambda_1}{2\lambda_2},\quad a_2=\frac{\lambda_4}{\lambda_2},\quad a_3=\frac{\lambda_2\lambda_3-\lambda_1\lambda_4}{\lambda_2^2}$$

$$a_4=\frac{2\lambda_2(\lambda_2-\lambda_4)-\lambda_1(\lambda_2\lambda_3-\lambda_1\lambda_4)}{\lambda_2^2\sqrt{\lambda_1^2-4\lambda_2}}$$

$$b_1=\frac{\sqrt{\lambda_1^2-4\lambda_2}}{2\lambda_2}, \quad (50)$$

and $\lambda_1^2-4\lambda_2>0$.

Applying the inverse Laplace transform to Eq. (49), we obtain

$$A(t)=a_2+a_3\cosh(b_1 t)\exp(-a_1 t)$$

$$+a_4\sinh(b_1 t)\exp(-a_1 t). \quad (51)$$

Now writing Eq. (48) in the following equivalent form

$$\overline{B}_n(\xi,q)=-\frac{1}{q_{1,n}(\xi)q_{2,n}(\xi)q_{3,n}(\xi)}$$

$$\times\frac{1}{q}[1+\frac{1}{q_{1,n}(\xi)}+\frac{1}{q_{2,n}(\xi)}+\frac{1}{q_{3,n}(\xi)}]$$

$$+\frac{\beta_{1,n}}{q-q_{1,n}(\xi)}+\frac{\beta_{2,n}}{q-q_{2,n}(\xi)}+\frac{\beta_{3,n}}{q-q_{3,n}(\xi)} \quad (52)$$

Where

$$\beta_{1,n}=$$

$$\frac{1}{(q_{1,n}(\xi))^2(q_{1,n}(\xi)-q_{2,n}(\xi))(q_{1,n}(\xi)-q_{3,n}(\xi))} \quad (53)$$

$$\beta_{2,n}=$$

$$\frac{1}{(q_{2,n}(\xi))^2(q_{2,n}(\xi)-q_{1,n}(\xi))(q_{2,n}(\xi)-q_{3,n}(\xi))} \quad (54)$$

$$\beta_{3,n}=$$

$$\frac{1}{(q_{3,n}(\xi))^2(q_{3,n}(\xi)-q_{1,n}(\xi))(q_{3,n}(\xi)-q_{2,n}(\xi))} \quad (55)$$

Applying inverse Laplace transform to Eq. (52), we obtain

$$B_n(\xi,t)=-\frac{1}{q_{1,n}(\xi)q_{2,n}(\xi)q_{3,n}(\xi)}$$

$$\times[1+\frac{1}{q_{1,n}(\xi)}+\frac{1}{q_{2,n}(\xi)}+\frac{1}{q_{3,n}(\xi)}+t]$$

$$+\beta_{1,n}\exp(q_{1,n}t)+\beta_{2,n}\exp(q_{2,n}t)$$

$$+\beta_{3,n}\exp(q_{3,n}t). \quad (56)$$

Let

$$\tau_n(\xi,t)=(A*B)(t)$$

$$=\int_0^t A(t-q)\,B_n(\xi,q)\,dq, \quad (57)$$

Using Eqs. (51) and (56) in the above equation, we obtain

$$\tau(n,t)=\int_0^t[a_2+a_3\cosh(b_1(t-q))\exp(-a_1(t-q))$$

$$+a_4\sinh(b_1(t-q))\exp(-a_1(t-q))]$$

$$\times[-\frac{1}{q_{1,n}(\xi)q_{2,n}(\xi)q_{3,n}(\xi)}$$

$$\times\{1+\frac{1}{q_{1,n}(\xi)}+\frac{1}{q_{2,n}(\xi)}+\frac{1}{q_{3,n}(\xi)}+q\}$$

$$+\beta_{1,n}\exp(q_{1,n}q)+\beta_{2,n}\exp(q_{2,n}q)$$

$$+\beta_{3,n}\exp(q_{3,n}q)]. \quad (58)$$

Finally, inversion of Eqs. (45) and (46) by means of Laplace transform and using Eq. (58), we obtain

$$\tau_1(x,y,t)=-\frac{2\nu}{d}\sum_{n=1}^{\infty}\sqrt{\frac{2}{\pi}}A_n(\xi)\sin(\lambda_n y)$$

$$\times\int_0^{\infty}\xi\sin(\xi x)[\tau_n(\xi,t)H(t)+2\sum_{p=1}^{\infty}(-1)^p$$

$$\times\tau_n(\xi,t-pT)H_{pT}(t)], \quad (59)$$

$$\tau_2(x,y,t)=\frac{2\nu}{d}\sum_{n=1}^{\infty}\sqrt{\frac{2}{\pi}}A_n(\xi)\cos(\lambda_n y)$$

$$\times\int_0^{\infty}\cos(\xi x)[\tau_n(\xi,t)H(t)+2\sum_{p=1}^{\infty}(-1)^p$$

$$\times\tau_n(\xi,t-pT)H_{pT}(t)]. \quad (60)$$

5. RESULTS AND DISCUSSIONS

The present problem is concerned with unsteady motion of generalized Burgers' fluid generated from rest induced by sawtooth pluses stress. Laplace transform technique and Fourier cosine and sine transforms have been used as mathematical tools in this order. The obtained expression for the velocity field has been written as the sum of Newtonian and non-Newtonian contributions.

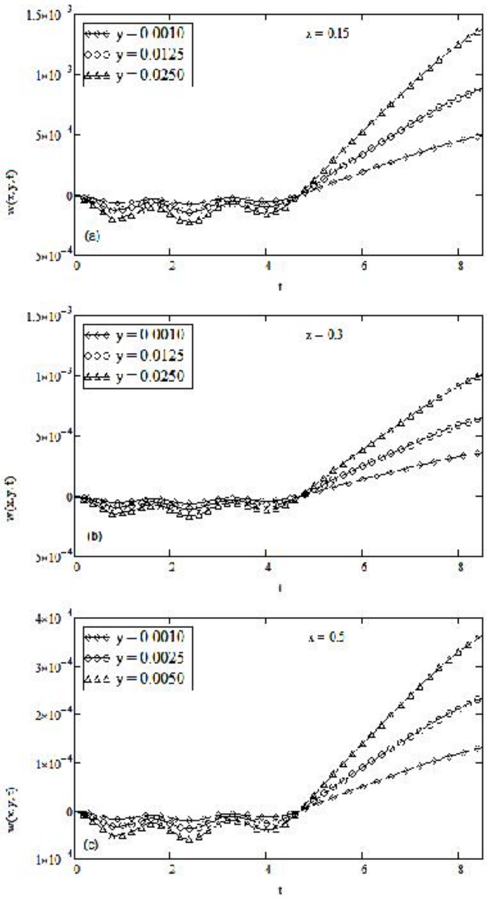

Fig. 1. Velocity profiles for generalized Burgers' fluid for different values of x **and** y **. Other parameters and values are taken as** $U = 15$, $d = 0.01$, « $= 3.9$, ¬ $= 0.004$, ª$_1 = 3$, ª$_2 = 2$, ª$_3 = 0.5$, ª$_4 = 2$, $\Omega = 0.5$, £ $= 0.3$.

By using the numerical calculations and graphical illustrations, the following physical aspects of the fluid behavior have been analyzed.

(a) Influence of side walls and pulse oscillation on the velocity field.

Fig. 1 represents the graphs for three values of distance from the side walls till the middle of the channel i.e., for different values of y for a fixed distance (d=0.01) between the side walls, and at different positions $x = 0.15, 0.3, 0.5$ from the bottom plate. It is seen that the pulse oscillations decrease far from the bottom plate. It is also observed that as the bottom plate is set into pulse oscillation, with respect to y-coordinate, the velocity increases from zero to a maximum till the middle of the channel. Moreover, it reveals that there is a time interval in which the motion is oscillatory, and then the oscillations of the velocity are attenuated.

Fig. 2 predicts the temporal velocity profile for various values of X and for three values of distances $d = 0.2, 0.4, 0.6$ between the walls. It is seen that as the distance d between the side walls increases, the magnitude of velocity profile also increases and the motion is oscillatory in some time interval, and then the pulse oscillations attenuated.

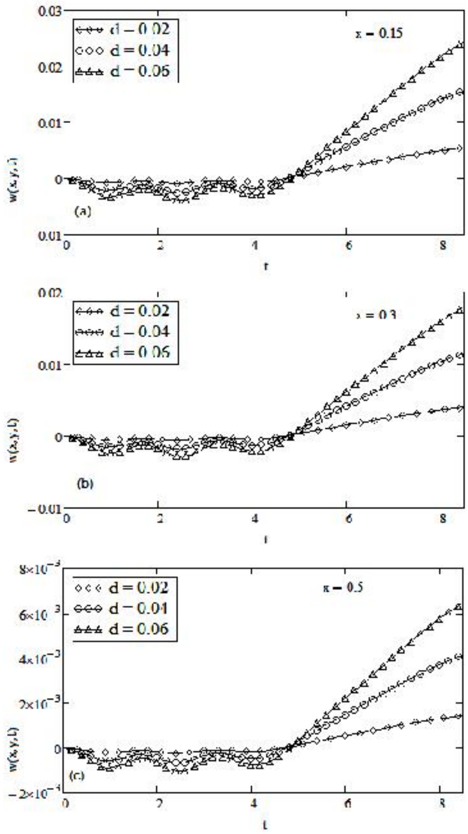

Fig. 2. Velocity profiles for generalized Burgers' fluid for different values of x **and** d **. Other parameters and values are taken as** $U = 15$, y=0.25 , « $= 3.9$, ¬ $= 0.004$, ª$_1 = 3$, ª$_2 = 2$, ª$_3 = 0.5$, ª$_4 = 2$, $\Omega = 0.5$, £ $= 0.3$.

(b) Effects of pulse period, magnetic and porosity parameters

In Fig. 3, the graphs for three values of pulse period $T = 2, 3, 4$ are presented, for different values of the magnetic field strength Ω verses y. As Ω increases, the magnitude of pulse oscillation also increases and the velocity profile reaches the steady state slower, there exist critical values of the magnetic field strength where its effect is negligible (in our case the minimum and maximum values are 0.01 and 100 respectively), while the pulse period T decreases the magnitude of pulse oscillations of velocity profile. The effect of £ represented in Fig. 4 is similar to that of Ω as represented in Fig.3.

In order to study the effect of the magnetic field strength and porosity on the velocity field verses t , we use a numerical procedure for the inverse Laplace transforms, namely the Stehfest's algorithm, (1972). If we denote by $W(x, y, q)$ the inverse Fourier transform of the function given by Eq. (21), then, in accordance with Stehfest's algorithm, the inverse Laplace transform, namely the velocity field w (x,y,t) is given by

$$w(x,y,t) = \frac{\ln 2}{t} \sum_{j=1}^{2s} d_j W\left(x,y,j\frac{\ln 2}{t}\right)$$

Where

$$d_j = (-1)^{j+s} \sum_{i=\left[\frac{j+1}{2}\right]}^{\min(j,s)} \frac{i^s (2i)!}{(s-i)!\, i!\, (i-1)!\, (j-i)!\, (2i-j)!}.$$

S is a positive integer number and $[r]$ denotes the integer part of the real number r.

The influence of the magnetic field on the velocity field is shown in Fig. 5. From this diagram, we observe that the influence of magnetic field is to decrease the magnitude of pulse oscillation and the velocity reaches the steady state earlier. Fig. 6 is drawn to show the effect of porosity on the fluid velocity. Note that, the effect of porosity is similar as that of the magnetic field.

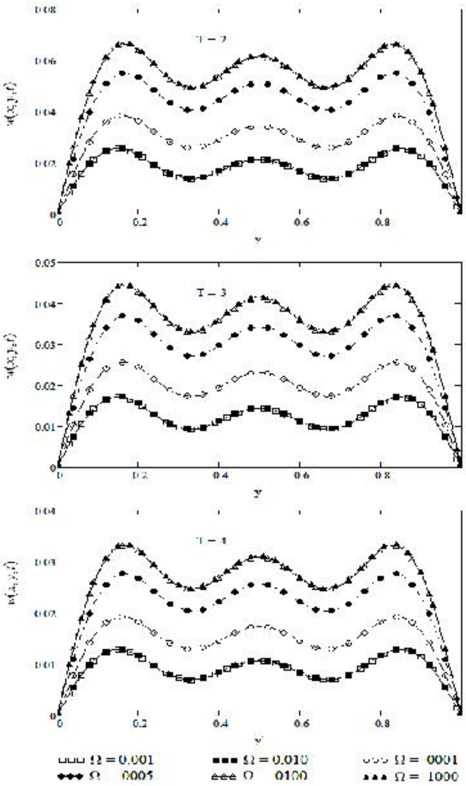

Fig. 3. Velocity profiles for generalized Burgers' fluid for different values of Ω and T . Other parameters and values are taken as U = 15 , t=1.5 , « = 3.9 , ¬ = 0.004 , a_1=5 , a_2 = 2 , a_3 = 0.5 , a_4=2 , d = 1 , £ = 0.3 .

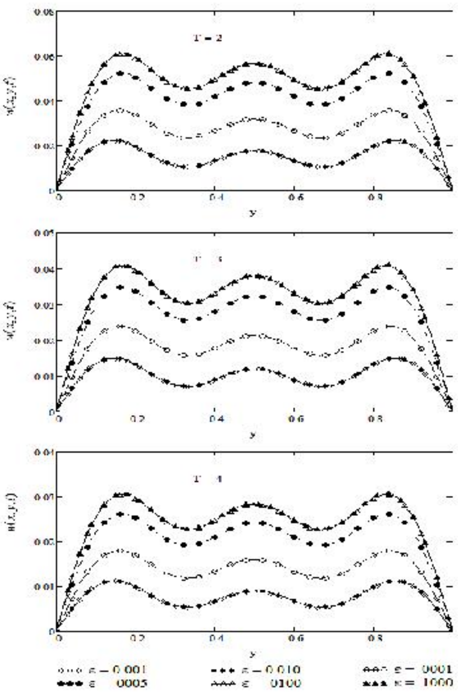

Fig. 4. Velocity profiles for generalized Burgers' fluid for different values of £ and T . Other parameters and values are taken as U = 15 , t=1.5 , « = 3.9 , ¬ = 0.004 , a_1=5 , a_2 = 2 , a_3 = 0.5 , a_4=2 , d = 1 , Ω = 0.1

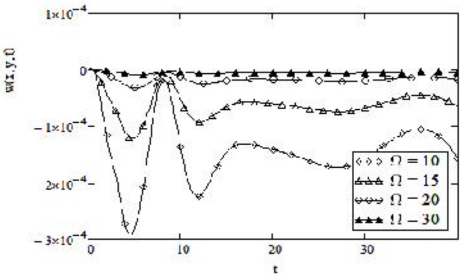

Fig. 5. Velocity profiles for generalized Burgers' fluid for different values of Ω . Other parameters and values are taken as U = 15 , x = 0.5 , y = 0.5 « = 3.9 , ¬ = 0.004 , a_1=2 , a_2 = 0.7 , a_3 = 0.3 , a_4=3 , d = 1 , £ = 0.3

(c) Influence of parameters a_1, a_2, a_3 and a_4 on the velocity field.

In order to see the effects of the material parameters a_1, a_2, a_3 and a_4, various values of these parameters are considered to plot the Figs. 7 and 8. In these Figs., we used the numerical values $U = 15$, $x = 0.01$, $y = 0.01$, $d = 0.02$, $T = \frac{®}{4}$, « = 3.9, ³ = 0.004 , £ = 0.3 , $\Omega = 1.7$ and three values for a_1, a_2, a_3 and a_4. In Fig. 7 the parameter a_1 and a_3 are variables and parameters a_2 and a_4 are constants. It can be seen that, if the value of the parameter a_1 increases, the fluid flows more slowly. Also, be noted that for the same value of the parameter a_1, and by increasing a_3 results in decreasing the velocity of fluid (the velocity amplitudes decrease if the values of a_3 increase). Fig. 8 corresponds to the variation of the parameters a_2 and a_4. There is no significant effect in the early period of the movement. The difference appears in the behavior of the fluid velocity, when compared with a_1, velocity amplitudes increase if the parameter a_2 increases. And in this case, for a constant value of the parameter a_2, and by increasing a_4. it is clear that, the fluid flows more faster.

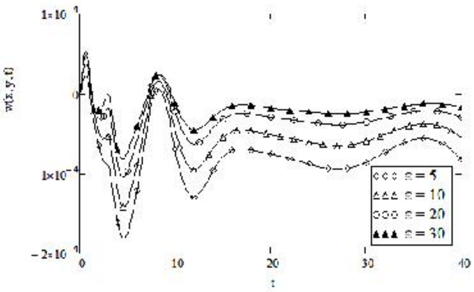

Fig. 6. Velocity profiles for generalized Burgers' fluid for different values of £ . Other parameters and values are taken as $U = 15$, $x = 0.5$, $y = 0.5$ $« = 3.9$, $¬ = 0.004$, $a_1 = 2$, $a_2 = 0.7$, $a_3 = 0.3$, $a_4 = 3$, $d = 1$, $\Omega = 10$.

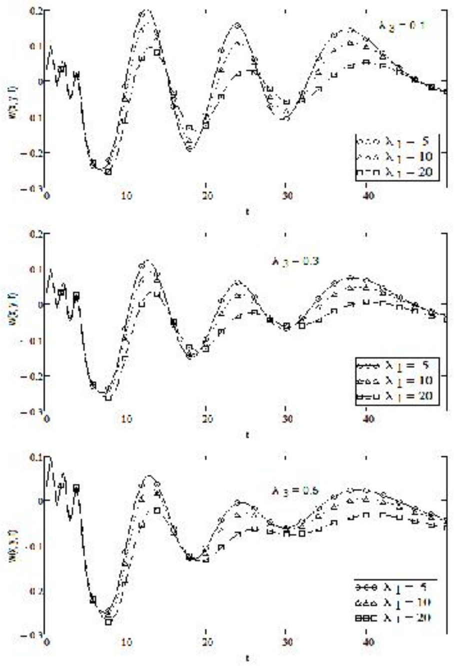

Fig. 7. Velocity profiles for generalized Burgers' fluid for different values of a_1 and a_3. Other parameters and values are taken as $U = 15$, $« = 3.9$, $¬ = 0.004$, x=0.01, y=0.01, $a_2 = 7$, $T = \frac{®}{4}$, $a_4 = 3$, $d = 0.02$, $\Omega = 1.7$, $£ = 0.3$.

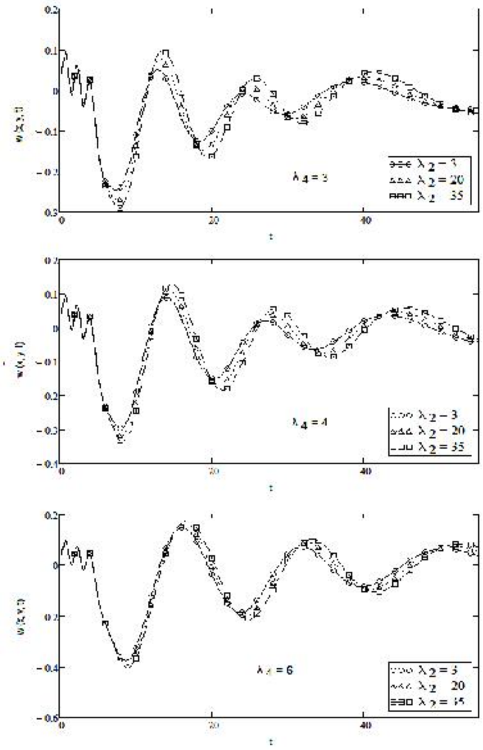

Fig. 8. Velocity profiles for generalized Burgers' fluid for different values of a_2 and a_4. Other parameters and values are taken as $U = 15$, $« = 3.9$, $¬ = 0.004$, x=0.01, y=0.01, $a_1 = 5$, $T = \frac{®}{4}$, $a_3 = 0.5$, $d = 0.02$, $\Omega = 1.7$ $£ = 0.3$.

6. CONCLUSSIONS

Here we obtained analytical solutions for the magnetohydrodynamic flow of a generalized Burgers' fluid between two parallel side walls. The expressions for the velocity field and the corresponding tangential stresses induced by the sawtooth pulses stress are obtained by means of the Laplace and Fourier cosine and sine transforms. The main findings are summarized as follows:

- The amplitude of pulse oscillation of velocity profile decreases far from the bottom plate.
- The magnitude of pulse oscillation of velocity profile increases from zero to maximum from the side walls till the middle of the channel.
- There is a time interval in which the velocity is oscillatory and then the oscillations of the velocity are attenuated.
- As distance between the walls increases, the magnitude of pulse oscillation of velocity profile also increases initially and then the oscillations of the velocity are attenuated.
- Increasing magnetic field strength and porosity, lead to decrease the amplitude of pulse oscillation verses t whereas the amplitudes increase verses y .

- As the values of the parameters a_1 and a_3 increase, the fluid flows more slowly whereas behavior of parameters a_2 and a_4 is opposite when compared to that of a_1 and a_3.

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