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Mathematical Model of Boundary Layer Flow over a Moving Plate in a Nanofluid with Viscous Dissipation

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ABSTRACT

In this study, the numerical investigation of boundary layer flow over a moving plate in a nanofluid with viscous dissipation and constant wall temperature is considered. The governing non-linear partial differential equations are first transformed into a system of ordinary differential equations using a similarity transformation. The transformed equations are then solved numerically using the Keller-box method. Numerical solutions are obtained for the Nusselt number, Sherwood number and the skin friction coefficient as well as the concentration and temperature profiles. The features of the flow and heat transfer characteristics for various values of the Prandtl number, plate velocity parameter, Brownian motion and thermopherosis parameters, Eckert number and Lewis number are analyzed and discussed. It is found that the presence of viscous dissipation reduces the range of the plate velocity parameter for which the solution exists. The increase of both Brownian motion and thermophoresis parameters results to the decrease of the Nusselt number, while the Sherwood number increases with the increase of the thermophoresis parameter.

Keywords: Constant wall temperature; Moving plate; Nanofluid; Viscous dissipation.

NOMENCLATURE

C	nanoparticle volume fraction
C_f	skin friction coefficient
C_{fr}	reduced skin friction coefficient
C_p	specific heat capacity at constant pressure
C_w	nanoparticle volume fraction at the surface
C_∞	ambient nanoparticle volume fraction
D_B	Brownian diffusion coefficient
D_T	thermophoresis diffusion coefficient
Ec	Eckert number
j_w	surface mass flux
k	thermal conductivity
Le	Lewis number
N_b	Brownian motion parameter
N_t	thermophoresis parameter
Nu_x	local Nusselt number
Nur	Reduced Nusselt number
Pr	Prandtl number
q_w	surface heat flux
Re_x	local Reynolds number
Shr	reduced Sherwood number
T	temperature inside the boundary layer
T_w	wall temperature
T_∞	ambient temperature
U_∞	free stream velocity
u	velocity components along the x-direction
u_w	plate velocity
v	velocity components along the y-direction
£	plate velocity parameter
¦	similarity variable
¦$_\infty$	boundary layer thickness
Δ¦	step size
—	dimensionless temperature of the fluid
«	dynamic viscosity
¬	kinematic viscosity
°	fluid density
²	ratio of the effective heat capacity of the nanoparticle material and the heat capacity of ordinary fluid.
²$_w$	surface shear stress

Sh_x	Sherwood number
¤	rescaled nanoparticle volume fraction
·	stream function

1. INTRODUCTION

Convection boundary layer flow plays an important role in engineering and industrial activities. These configurations are applied in thermal effects managements in many industrial outputs for example in electronic devices, computer power supply and also in engine cooling system such as heat sink in car radiator. Because of the large contributions, this topic has attracted many researchers to study and expand the knowledge so that it could be applied in order to handle the thermal problems produced by these industrial outputs (Pop and Ingham 2001; Salleh *et al.* 2010).

The study of boundary layer flow on a constant speed moving plate was first studied by Sakiadis (1961). Due to entrainment of the ambient fluid, this boundary layer flow is quite different from Blasius flow past a flat plate. Sakiadis' theoretical predictions for Newtonian fluids were later corroborated experimentally by Tsou *et al.* (1967). Karwe and Jaluria (1988) considered the mixed convection from a moving plate in rolling and extrusion processes. Other papers that considered the boundary layer flow over a moving plate are Kumari and Nath (1996), Ali and Al-Yousef (1998), Chen (1999) and Elbashbeshy and Bazid (2000) which introduced the force convection, impulsive motion, suction or injection effects and temperature dependent viscosity, respectively. Bataller (2008) and Ishak *et al.* (2011) observed the radiation effects on the thermal boundary layer flow for Blasius and Sakiadis flows with convective boundary conditions, respectively. It was found that the presence of thermal radiation and convective boundary conditions reduce the heat transfer rate at the surface. Next, the effect of transpiration on the flow and self-similar boundary layer flow over a moving surface was studied by Weidman *et al.* (2006) and Ishak *et al.* (2009). A permeable surface was considered and it was found that dual solutions were obtained in both studies. Weidman *et al.* (2006) reported that the solution is unique for positive values of the plate velocity parameter, while dual solutions exist for its negative values, up to the critical value, beyond which no solution is in existence. The stability analysis revealed that the upper branch solution is stable, while the lower branch solution is unstable.

The investigations involving the flow on a moving plate were also extended to other type of fluids such as viscoelastic fluid, micropolar fluid and nanofluid by many investigators including Abel *et al.* (2005), Anuradha and Krishnambal (2010) and recently by Khan *et al.* (2014) who considered the flow in a viscoelastic fluid. The MHD and buoyancy effects on a moving stretching surface were analysed and discussed. Aman *et al.* (2013) and Kundu *et al.* (2015) investigated the MHD effects in micropolar fluid while Sandeep and Sugunamma (2014) considered the unsteady hydromagnetic flow past an impulsively moving vertical plate in a porous medium. Furthermore, Bachok *et al.* (2010) and Roşca and Pop (2014) investigated the steady and unsteady boundary layer flow of a nanofluid past a moving surface in an external uniform free stream, respectively. Both studies considered the Bongiorno-Darcy nanofluid model. The effects of the Prandtl number, plate velocity parameter and the nanofluid parameter which is Brownian motion parameter, thermophoresis parameter and Lewis number was analysed and discussed. The numerical solutions were obtained by using the Keller-box method and the bvp4c package in Matlab, respectively.

In considering the viscous dissipation effects, from literature study it is found that Gebhart (1962) is probably the first who studied viscous dissipation in free convection flow. The viscous dissipation effects on unsteady free convective flow over a vertical porous plate was then investigated by Soundalgekar (1972). Vajravelu and Hadjinicolaou (1993) then studied the viscous dissipation effects on the flow and heat transfer over a stretching sheet. Chen (2004) observed the heat and mass transfer in MHD free convection from a vertical surface with Ohmic heating and viscous dissipation effects. Next, Partha *et al.* (2005) and Yirga and Shankar (2013) considered this topic on exponentially stretching surface and stagnation point flow in a nanofluid with thermal radiation and magnetohydrodynamic effects, respectively. It was reported that the presence of viscous dissipation which represented by Eckert number has contributed to the increase in velocity and thermal boundary layer thicknesses while the local Nusselt number decreases. Recently, Pal and Mondal (2014) and Gangadhar (2015) investigated the viscous disipation efects on a stretching plate, Blasius and Sakiadis plate with Soret-Dufour effects, radiation, heat generation, MHD, porous medium and Ohmic heating, respectively.

Motivated by the above factor and contributions, the objective of the present paper is to investigate the boundary layer flow over a moving flat plate in a nanofluid with viscous dissipation. It is known that nanofluid is a fluid containing nanometer-sized particles, called nanoparticles. This type of fluid is believed may enhance thermal conductivity, viscosity, thermal diffusivity and convective heat transfer compared to those base fluids like water and oil. Many investigations on the nanofluid have been done such as Vajravelu *et al.* (2011), Anwar *et al.* (2012), Bachok *et al.* (2012) and recently by Kameswaran *et al.* (2013), Rashad *et al.* (2013), Makinde *et al.* (2013) and Sreenivasulu and Bhaskar (2015). To the best of our knowledge, the present study is never been considered before, so that the reported results are new.

2. MATHEMATICAL FORMULATIONS

A steady two-dimensional boundary layer flow over a moving plate immersed in a nanofluid of ambient temperature T_∞ is considered. It is assumed that T be the temperature inside the boundary layer, T_w be the wall temperature, U_∞ be the free stream velocity and $u_w(x)=\lambda U_\infty$ be the plate velocity where λ is the plate velocity parameter (see Weidman *et al.* 2006). Furthermore, C is the nanoparticle volume fraction, C_w is the nanoparticle volume fraction at the surface and C_∞ is the ambient nanoparticle volume fraction The physical model and coordinate system of this problem is shown in Fig. 1. The governing boundary layer equations are:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho C_p}\frac{\partial^2 T}{\partial y^2}+\tau\left[D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y}+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right]+\frac{\mu}{\rho C_p}\left(\frac{\partial u}{\partial y}\right)^2, \tag{3}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D_B\frac{\partial^2 C}{\partial y^2}+\frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2}, \tag{4}$$

subject to the boundary conditions

$$u=u_w(x)=\lambda U_\infty,\ v=0,\ T=T_w,\ C=C_w \text{ at } y=0,$$
$$u=U_\infty,\ T\to T_\infty,\ C\to C_\infty \text{ as } y\to\infty, \tag{5}$$

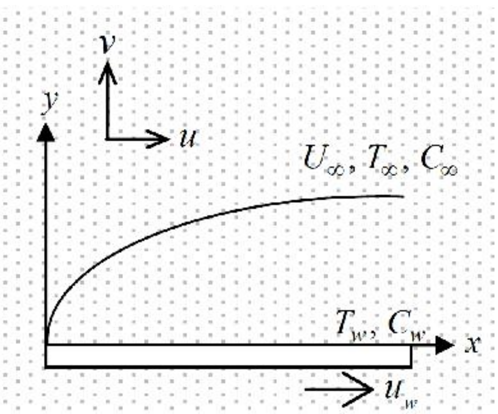

Fig. 1. Physical model of the coordinate system.

where u and v are the velocity components along the x and y directions, respectively, μ is the dynamic viscosity, ν is the kinematic viscosity, ρ is the fluid density, k is the thermal conductivity and C_p is the specific heat capacity at constant pressure. Next, D_B is the Brownian diffusion coefficient, D_T is the thermophoresis diffusion coefficient, τ is the ratio of the effective heat capacity of the nanoparticle material and the heat capacity of the ordinary fluid.

The set of similarity transformation for Eqs. (1) to (4) subjected to the boundary conditions (5) is (Weidman *et al.* 2006; Bachok *et al.* 2010)

$$\eta=\left(\frac{U_\infty}{2\nu x}\right)^{1/2}y,\ \psi=\left(2U_\infty\nu x\right)^{1/2}f(\eta),$$
$$\theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty},\ \phi(\eta)=\frac{C-C_\infty}{C_w-C_\infty}, \tag{6}$$

where ϕ and θ are the rescaled nanoparticle volume fraction and dimensionless temperature of the fluid, respectively. ψ is the stream function defined as $u=\frac{\partial\psi}{\partial y}$ and $v=-\frac{\partial\psi}{\partial x}$ which identically satisfy Eq. (1). Then, u and v can be derived as

$$u=U_\infty f'(\eta),$$
$$v=-\left(\frac{U_\infty\nu}{2x}\right)^{1/2}f(\eta)+\frac{U_\infty y}{2x}f'(\eta), \tag{7}$$

By substituting Eqs. (6) and (7) into Eqs. (2) to (4), we have

$$f'''+ff''=0 \tag{8}$$

$$\frac{1}{\Pr}\theta''+f\theta'+N_b\theta'\phi'+N_t\theta'^2+Ecf''^2=0, \tag{9}$$

$$\phi''+\frac{N_t}{N_b}\theta''+Lef\phi'=0 \tag{10}$$

Where $\Pr=\frac{\nu\rho C_p}{k}$ is the Prandtl number, $N_b=\frac{\tau D_B(C_w-C_\infty)}{\nu}$ is the Brownian motion parameter, $N_t=\frac{\tau D_T(T_w-T_\infty)}{T_\infty\nu}$ is the thermophoresis parameters, $Ec=\frac{(U_\infty)^2}{C_p(T_w-T_\infty)}$ is the Eckert number and $Le=\frac{\nu}{D_B}$ is the Lewis number.

The boundary conditions (5) become

$$f(0)=0,\ f'(0)=\lambda,\ \theta(0)=1,\ \phi(0)=1,$$
$$f'(\eta)\to 1,\ \theta(\eta)\to 0,\ \phi(\eta)\to 0, \text{ as } y\to\infty \tag{11}$$

Note that $\lambda>0$ corresponds to downstream movement of the plate from the origin (Weidman *et al.* 2006). The physical quantities of interest are the skin friction coefficient C_f the local Nusselt number Nu_x and the local Sherwood number Sh_x

which are given by

$$C_f = \frac{\tau_w}{\rho u_e^2},\ Nu_x = \frac{x q_w}{k(T_w - T_\infty)},\quad Sh_x = \frac{x j_w}{D_B(C_w - C_\infty)}. \tag{12}$$

The surface shear stress τ_w, the surface heat flux q_w and the surface mass flux j_w are given by

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0},\ q_w = -k\left(\frac{\partial T}{\partial y}\right)_{y=0},\quad j_w = -D_B\left(\frac{\partial C}{\partial y}\right)_{y=0} \tag{13}$$

with $\mu = \rho\nu$ being the dynamic viscosity. Using the similarity variables in Eq. (6) give

$$C_f\left(2\mathrm{Re}_x\right)^{1/2} = f''(0),\ Nu_x\left(\frac{\mathrm{Re}_x}{2}\right)^{-1/2} = -\theta'(0),\quad Sh_x\left(\frac{\mathrm{Re}_x}{2}\right)^{-1/2} = -\phi'(0) \tag{14}$$

where $\mathrm{Re}_x = \frac{U_\infty x}{\nu}$ is the local Reynolds number. It is worth mentioning that the physical quantities of interest in the present context are

$$C_f\left(2\mathrm{Re}_x\right)^{1/2},\ Nu_x\left(\frac{\mathrm{Re}_x}{2}\right)^{-1/2} \text{ and } Sh_x\left(\frac{\mathrm{Re}_x}{2}\right)^{-1/2}$$

which are referred as the reduced skin friction coefficient, the reduced Nusselt number and the reduced Sherwood number and can be denoted as C_{fr}, Nur and Shr which are represented by $f''(0)$, $-\theta'(0)$ and $-\phi'(0)$, respectively.

3. NUMERICAL METHOD

The transformed ordinary differential equations subject to the associated boundary conditions are solved numerically using the Keller-box method. As described in the books by Na (1979) and Cebeci and Bradshaw (1988), the solution is obtained in the following four steps: Firstly, reduce Eqs. (8) to (11) to a first-order system. Secondly, write the difference equations using central differences; Thirdly, linearize the resulting algebraic equations by Newton's method, and write them in the matrix-vector form; and finally, solve the linear system by the block tridiagonal elimination technique.

4. RESULTS AND DISCUSSION

Six parameters are considered, namely the Prandtl number $\Pr$, the plate velocity parameter λ, the Brownian motion parameter N_b, the thermophoresis parameter N_t, the Eckert number Ec, and the Lewis number Le. The step size $\Delta\eta = 0.02$ and the boundary layer thickness η_∞ from 2.5 to 8 were used in obtaining the numerical results. From the literature review, in considering the boundary layer flow over a stretching surface in nanofluids, it was found that N_b and N_t effects are practically studied at range between 0.1 and 0.5 while Le in the range of 1 to 40 (Khan and Pop 2010; Noghrehabadi *et al.* 2012; Nandy and Mahapatra 2013). Therefore, in the present study, the same range is considered in our analysis and discussion.

Table 1 shows the comparison values of $-\theta'(0)/\sqrt{2}$ with previous results by Roşca and Pop (2014) for various values of Prandtl number $\Pr$. It is found that they are in good agreement. We can conclude that this method works efficiently for the present problem.

Table 2 shows the values of Nur, Shr and C_{fr} for various values of λ. From this table, it is found that the present of plate velocity parameter λ results to the increase of Nur and Shr. Meanwhile, the skin friction coefficient C_{fr} decreases as λ increases. Note that when $\lambda = 0$, $C_{fr} = 0.4696$ which is in a very good agreement with Blasius (1908). Next, $C_{fr} = 0$ for $\lambda = 1$ due to the plate and the fluid move with the same velocity which results in no velocity gradient, i.e. there is no friction at the fluid-solid interface.

Table 1 Comparison values of $-\theta'(0)/\sqrt{2}$ with previous published results for various values of Pr when $\lambda = N_b = N_t = Ec = Le = 0$

Pr	Roşca and Pop (2014)	Present
0.7	0.29268	0.292680
0.8	0.30691	0.306917
1	0.33205	0.332057
5	057668	0.576689
10	0.72814	0.728141

Table 2. Values of Nur, Shr and C_{fr} for various values of λ when $\Pr = 7$, $N_b = N_t = Ec = 0.1$ and $Le = 10$

λ	Nur	Shr	C_{fr}
0	0.3747	1.1672	0.4696
0.1	0.4705	1.3369	0.4625
0.5	0.7875	1.8657	0.3288
1	1.0717	2.3805	0
2	1.2994	3.3099	-1.0191

Table 3 presents the values of the reduced Nusselt number Nur and Sherwood number Shr for various values of N_b and N_t. From this table, it is concluded that the increase of both parameters N_b and N_t results to the decrease of Nur while Shr increases with the increase of N_t. Physically, it is suggested that the small values of N_b and N_t enhance convective heat transfer capabilities while the large values of N_b and N_t enhance the convective mass transfer capabilities.

Figures 2 to 4 illustrate the temperature profiles for various values of N_b, N_t, Le, Ec and ε. These Figs are plotted in order to understand the behavior of the thermal boundary layer thickness for various values of parameter discussed. From Fig. 2, it is found that as N_b increases, the thermal boundary layer thickness also increases. According to Anwar *et al.* (2013), this is due to the fact that larger values of N_b have a large extent of fluid hence thickening the thermal boundary layer thickness. The same trends happen for parameter N_t. This result from the deeper penetration into the fluid which cause to the thickening of thermal boundary layer thickness.

Table 3 Values of Nur and Shr for various values of N_b and N_t when $\Pr = 7$, $Ec = 0.1$, $Le = 10$ and $\varepsilon = 0.5$

N_b	N_t	Nur	Shr
0.1	0.1	0.7875	1.8657
0.2	0.1	0.4923	1.9951
0.3	0.1	0.2959	2.0091
0.4	0.1	0.1712	2.0010
0.5	0.1	0.0952	1.9884
0.1	0.2	0.6047	2.0601
0.1	0.3	0.4708	2.3624
0.1	0.4	0.3722	2.6994
0.1	0.5	0.2990	3.0319

In Fig. 3, it is found that the increase of Le results to the decrease in thermal boundary layer thickness. The opposite trend occurs for Ec where the increase of this parameter results to the increase of thermal boundary layer thickness. This may be explained as follows: The increase of Ec directly proportional to the increase of external velocity (see definition of Ec) and this situation reduces the effect of the plate velocity parameter ε, which thickening the thermal boundary layer thickness.

Figure 4 shows the temperature profiles for various values of ε. It is found that the present of plate velocity parameter $(\varepsilon > 0)$ reduces the thermal boundary layer thickness. Physically, the increase of ε results to the increase of ratio velocity differences between the plate and the fluid which enhance the fluid to move away from the plate region rapidly. This situation reduces the thermal diffusivity and thinning the thermal boundary layer thicknesses.

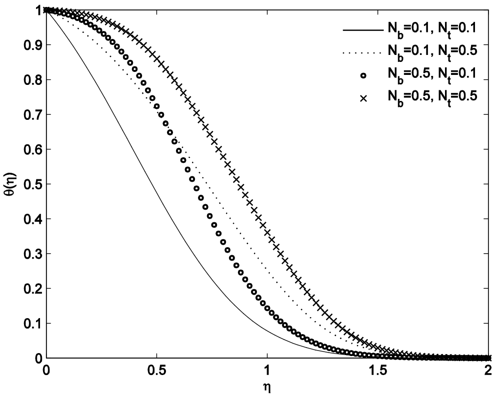

Fig. 2. Temperature profiles $\theta(\eta)$ for various values of N_b and N_t when $\Pr = 7$, $Ec = 0.1$, $Le = 10$ and $\varepsilon = 0.5$.

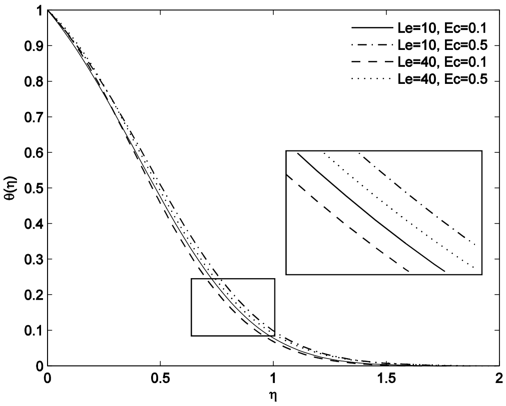

Fig. 3. Temperature profiles $\theta(\eta)$ for various values of Le and Ec when $N_b = N_t = 0.1$, $\Pr = 7$ and $\varepsilon = 0.5$.

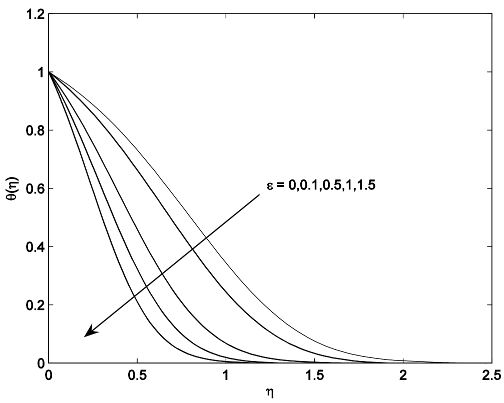

Fig. 4. Temperature profiles $\theta(\eta)$ for various values of ε when $N_b = N_t = 0.1$, $\Pr = 7$, $Le = 10$ and $Ec = 0.1$.

Figures 5 to 7 show the variation of nanoparticle volume fraction $\phi(\eta)$ for various values of parameters. The increase of N_b in Fig. 5 results to the decrease of $\phi(\eta)$. Also, the effects of ε is similar as in Fig. 4. The moving effects reduces the nanoparticle volume fraction $\phi(\eta)$.

The effects of N_t and Le are shown in Fig. 6. Both parameters consume the contra trend where the increase of Le is to the decrease the nanoparticle volume fraction $\phi(\eta)$ while $\phi(\eta)$ increases as N_t increases. Further, the nanoparticle volume fraction $\phi(\eta)$ increases monotonically as $\Pr$ decreases. Meanwhile, the Eckert number Ec shows the contradict trends. Large values of Ec means large viscous dissipation effect and this enhance the nanoparticle volume fraction $\phi(\eta)$. The variations of the nanoparticle volume fraction for both parameters $\Pr$ and Ec are plotted in Fig. 7.

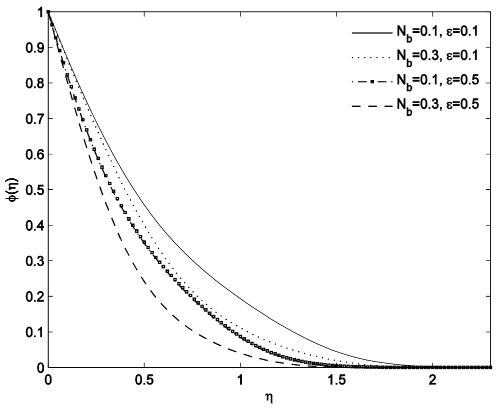

Fig. 5. Variation of nanoparticle volume fraction $\phi(\eta)$ for various values of N_b and ε when $\Pr = 7$, $N_t = Ec = 0.1$ and $Le = 10$.

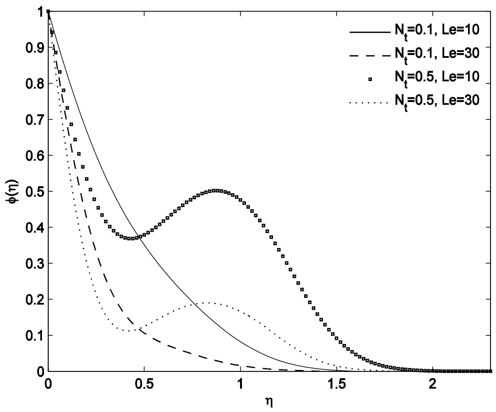

Fig. 6. Variation of nanoparticle volume fraction $\phi(\eta)$ for different values of N_t and Le when $\Pr = 7$, $N_b = Ec = 0.1$ and $\varepsilon = 0.5$.

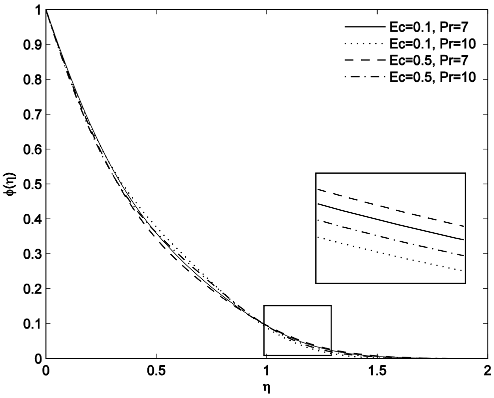

Fig. 7. Variation of nanoparticle volume fraction $\phi(\eta)$ for different values of $\Pr$ and Ec when $N_b = N_t = 0.1$, $Le = 10$ and $\varepsilon = 0.5$.

Figures 8 and 9 illustrate the variation of the reduced Nusselt number Nur and the reduced Sherwood number Shr with ε for various values of Ec. From Fig. 8, it is found that the presence of viscous dissipation $(Ec \neq 0)$ changes the variation of Nur curve to a quadratic curve. Also, the presence of Ec reduces the range of ε for which the solution exists It is clearly shown when $Ec = 0.1$, the physical acceptable solution occur until $\varepsilon = 4.0189$ while when the value of Ec increases $(Ec = 0.3)$, Nur stops at $\varepsilon = 2.7153$. Since $Nu \cong 0$, this means that there is no convection occur beyond this value, which leads to pure conduction heat transfer. Furthermore, it is noticed that when $\varepsilon = 1$, which result from the equivalent of stretching and external velocity at the plate surface, Ec does not effects the values of Nur.

In Fig. 9, it is noticed that the physical acceptable solution for ε has a similar trends as in Fig. 8. Further, it is found that, at large values of ε, larger values of Ec produce large values of Shr which physically means large in convective mass transfer. This is due to the increase of ratio of kinetic energy over enthalpy in increasing of Ec. Also, it is noticed that the variation of Shr is unique for all Ec when $\varepsilon = 1$ as in Fig. 8.

Figures 10 and 11 show the variation of Nur and Shr with $\Pr$ for various values of Le. From Fig. 10, it is found that small values of $\Pr$ such as liquid metal $(\Pr \ll 1)$, produce very small value of Nur which imply no convection occurs or the heat transfer is in pure conduction situation. It is realistic since liquid metals have high thermal conductivity but low in viscosity. For fixed values of $\Pr$, increasing Le results to the decrease of Nur but increase of Shr.

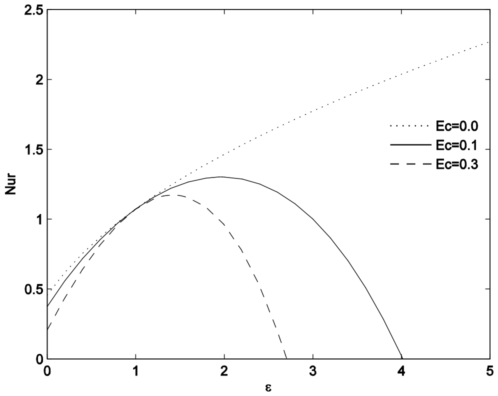

Fig. 8. Variation of reduced Nusselt number with ε for various values of Ec when $\Pr = 7$, $Le = 10$ and $N_b = N_t = 0.1$.

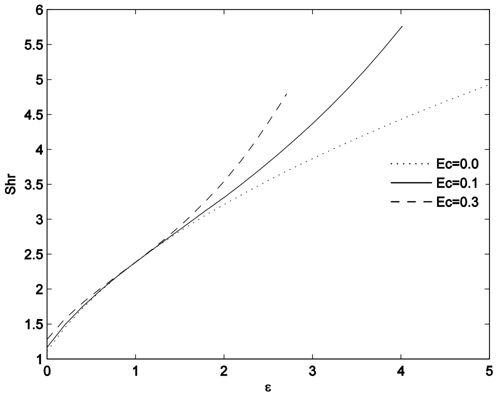

Fig. 9. Variation of reduced Sherwood number with ε for various values of Ec when $\Pr = 7$, $Le = 10$ and $N_b = N_t = 0.1$.

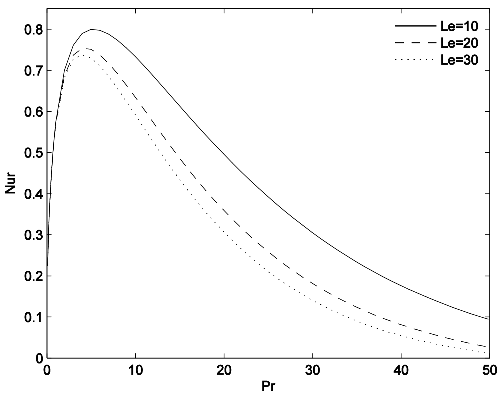

Fig. 10. Variation of reduced Nusselt number with Pr for various values of Le when $N_b = N_t = Ec = 0.1$ and $\varepsilon = 0.5$.

Lastly, Fig. 12 presents the variation of the reduced skin friction coefficient C_{fr} for various values of ε which produce $f'(0) = \varepsilon$ and $f'(\eta) = 1$ as $\eta \to \infty$. From this figure, it is found that the value of C_{fr} is positive for $\varepsilon < 1$, and C_{fr} decreases to 0 as ε approach 1. The value $C_{fr} = 0$ at $\varepsilon = 1$ is due to the plate moves in the same velocity with the fluid. Further, from this figure, we can conclude that C_{fr} decreases as ε increases.

From this study it is worth mentioning that only moving parameter affects the velocity profiles and the value of the skin friction coefficient. It is clear from the ordinary differential Eqs. (8) to (10) and boundary conditions (11).

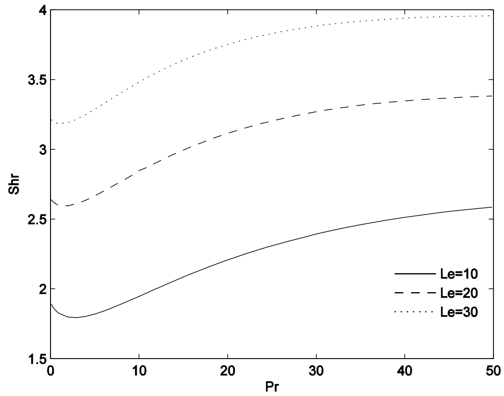

Fig. 11.Variation of reduced Sherwood number with Pr for various values of Le when $N_b = N_t = Ec = 0.1$ and $\varepsilon = 0.5$.

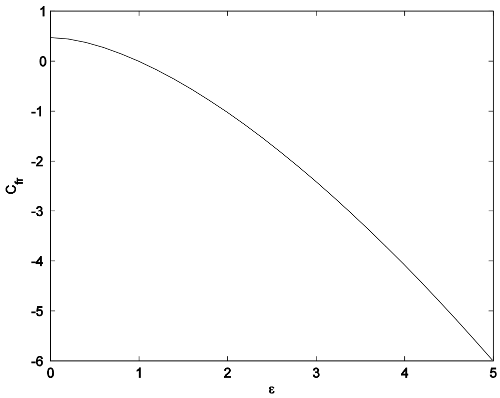

Fig. 12. Reduced skin friction coefficient for various values of ε.

5. CONCLUSION

In this paper, the boundary layer flow over a moving plate in a nanofluid in the presence of viscous dissipation was numerically studied. It was shown how the Prandtl number Pr, plate velocity parameter ε, Brownian motion parameter N_b, thermophoresis parameter N_t, Eckert number Ec and Lewis number Le affect the values of the reduced Nusselt and Sherwood numbers and the skin friction coefficient as well as the concentration and temperature profiles. As a conclusion, the thermal boundary layer thickness depends strongly on these parameters. It was found that in the presence of viscous dissipation, the range of the plate velocity parameter for which the solution exists reduces, which physically leads to pure conduction to occur. Further, the same situation occurs to the nanofluid with low Prandtl number. This may be explained as the low Prandtl value means the fluid is low in viscosity but high in thermal conductivity such as liquid metal. The increase of N_b, N_t and Ec results to the increase in thermal boundary layer thickness. Meanwhile, the increase of $\Pr, \varepsilon$ and Le reduced the thermal boundary layer thickness.

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Mathematical Model of Boundary Layer Flow over a Moving Plate in a Nanofluid with Viscous Dissipation

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ABSTRACT

In this study, the numerical investigation of boundary layer flow over a moving plate in a nanofluid with viscous dissipation and constant wall temperature is considered. The governing non-linear partial differential equations are first transformed into a system of ordinary differential equations using a similarity transformation. The transformed equations are then solved numerically using the Keller-box method. Numerical solutions are obtained for the Nusselt number, Sherwood number and the skin friction coefficient as well as the concentration and temperature profiles. The features of the flow and heat transfer characteristics for various values of the Prandtl number, plate velocity parameter, Brownian motion and thermopherosis parameters, Eckert number and Lewis number are analyzed and discussed. It is found that the presence of viscous dissipation reduces the range of the plate velocity parameter for which the solution exists. The increase of both Brownian motion and thermophoresis parameters results to the decrease of the Nusselt number, while the Sherwood number increases with the increase of the thermophoresis parameter.

Keywords: Constant wall temperature; Moving plate; Nanofluid; Viscous dissipation.

NOMENCLATURE

C	nanoparticle volume fraction	Shr	reduced Sherwood number
C_f	skin friction coefficient	T	temperature inside the boundary layer
C_{fr}	reduced skin friction coefficient	T_w	wall temperature
C_p	specific heat capacity at constant pressure	T_∞	ambient temperature
C_w	nanoparticle volume fraction at the surface	U_∞	free stream velocity
C_∞	ambient nanoparticle volume fraction	u	velocity components along the x-direction
D_B	Brownian diffusion coefficient	u_w	plate velocity
D_T	thermophoresis diffusion coefficient	v	velocity components along the y-direction
Ec	Eckert number	£	plate velocity parameter
j_w	surface mass flux	¦	similarity variable
k	thermal conductivity	¦ ∞	boundary layer thickness
Le	Lewis number	Δ¦	step size
N_b	Brownian motion parameter	¯	dimensionless temperature of the fluid
N_t	thermophoresis parameter	«	dynamic viscosity
Nu_x	local Nusselt number	¬	kinematic viscosity
Nur	Reduced Nusselt number	°	fluid density
Pr	Prandtl number	²	ratio of the effective heat capacity of the nanoparticle material and the heat capacity of ordinary fluid.
q_w	surface heat flux	² $_w$	surface shear stress
Re_x	local Reynolds number		

Sh_x	Sherwood number
¤	rescaled nanoparticle volume fraction
·	stream function

1. INTRODUCTION

Convection boundary layer flow plays an important role in engineering and industrial activities. These configurations are applied in thermal effects managements in many industrial outputs for example in electronic devices, computer power supply and also in engine cooling system such as heat sink in car radiator. Because of the large contributions, this topic has attracted many researchers to study and expand the knowledge so that it could be applied in order to handle the thermal problems produced by these industrial outputs (Pop and Ingham 2001; Salleh *et al.* 2010).

The study of boundary layer flow on a constant speed moving plate was first studied by Sakiadis (1961). Due to entrainment of the ambient fluid, this boundary layer flow is quite different from Blasius flow past a flat plate. Sakiadis' theoretical predictions for Newtonian fluids were later corroborated experimentally by Tsou *et al.* (1967). Karwe and Jaluria (1988) considered the mixed convection from a moving plate in rolling and extrusion processes. Other papers that considered the boundary layer flow over a moving plate are Kumari and Nath (1996), Ali and Al-Yousef (1998), Chen (1999) and Elbashbeshy and Bazid (2000) which introduced the force convection, impulsive motion, suction or injection effects and temperature dependent viscosity, respectively. Bataller (2008) and Ishak *et al.* (2011) observed the radiation effects on the thermal boundary layer flow for Blasius and Sakiadis flows with convective boundary conditions, respectively. It was found that the presence of thermal radiation and convective boundary conditions reduce the heat transfer rate at the surface. Next, the effect of transpiration on the flow and self-similar boundary layer flow over a moving surface was studied by Weidman *et al.* (2006) and Ishak *et al.* (2009). A permeable surface was considered and it was found that dual solutions were obtained in both studies. Weidman *et al.* (2006) reported that the solution is unique for positive values of the plate velocity parameter, while dual solutions exist for its negative values, up to the critical value, beyond which no solution is in existence. The stability analysis revealed that the upper branch solution is stable, while the lower branch solution is unstable.

The investigations involving the flow on a moving plate were also extended to other type of fluids such as viscoelastic fluid, micropolar fluid and nanofluid by many investigators including Abel *et al.* (2005), Anuradha and Krishnambal (2010) and recently by Khan *et al.* (2014) who considered the flow in a viscoelastic fluid. The MHD and buoyancy effects on a moving stretching surface were analysed and discussed. Aman *et al.* (2013) and Kundu *et al.* (2015) investigated the MHD effects in micropolar fluid while Sandeep and Sugunamma (2014) considered the unsteady hydromagnetic flow past an impulsively moving vertical plate in a porous medium. Furthermore, Bachok *et al.* (2010) and Roşca and Pop (2014) investigated the steady and unsteady boundary layer flow of a nanofluid past a moving surface in an external uniform free stream, respectively. Both studies considered the Bongiorno-Darcy nanofluid model. The effects of the Prandtl number, plate velocity parameter and the nanofluid parameter which is Brownian motion parameter, thermophoresis parameter and Lewis number was analysed and discussed. The numerical solutions were obtained by using the Keller-box method and the bvp4c package in Matlab, respectively.

In considering the viscous dissipation effects, from literature study it is found that Gebhart (1962) is probably the first who studied viscous dissipation in free convection flow. The viscous dissipation effects on unsteady free convective flow over a vertical porous plate was then investigated by Soundalgekar (1972). Vajravelu and Hadjinicolaou (1993) then studied the viscous dissipation effects on the flow and heat transfer over a stretching sheet. Chen (2004) observed the heat and mass transfer in MHD free convection from a vertical surface with Ohmic heating and viscous dissipation effects. Next, Partha *et al.* (2005) and Yirga and Shankar (2013) considered this topic on exponentially stretching surface and stagnation point flow in a nanofluid with thermal radiation and magnetohydrodynamic effects, respectively. It was reported that the presence of viscous dissipation which represented by Eckert number has contributed to the increase in velocity and thermal boundary layer thicknesses while the local Nusselt number decreases. Recently, Pal and Mondal (2014) and Gangadhar (2015) investigated the viscous disipation efects on a stretching plate, Blasius and Sakiadis plate with Soret-Dufour effects, radiation, heat generation, MHD, porous medium and Ohmic heating, respectively.

Motivated by the above factor and contributions, the objective of the present paper is to investigate the boundary layer flow over a moving flat plate in a nanofluid with viscous dissipation. It is known that nanofluid is a fluid containing nanometer-sized particles, called nanoparticles. This type of fluid is believed may enhance thermal conductivity, viscosity, thermal diffusivity and convective heat transfer compared to those base fluids like water and oil. Many investigations on the nanofluid have been done such as Vajravelu *et al.* (2011), Anwar *et al.* (2012), Bachok *et al.* (2012) and recently by Kameswaran *et al.* (2013), Rashad *et al.* (2013), Makinde *et al.* (2013) and Sreenivasulu and Bhaskar (2015). To the best of our knowledge, the present study is never been considered before, so that the reported results are new.

2. MATHEMATICAL FORMULATIONS

A steady two-dimensional boundary layer flow over a moving plate immersed in a nanofluid of ambient temperature T_∞ is considered. It is assumed that T be the temperature inside the boundary layer, T_w be the wall temperature, U_∞ be the free stream velocity and $u_w(x)=\lambda U_\infty$ be the plate velocity where λ is the plate velocity parameter (see Weidman *et al.* 2006). Furthermore, C is the nanoparticle volume fraction, C_w is the nanoparticle volume fraction at the surface and C_∞ is the ambient nanoparticle volume fraction The physical model and coordinate system of this problem is shown in Fig. 1. The governing boundary layer equations are:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho C_p}\frac{\partial^2 T}{\partial y^2}+\tau\left[D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y}+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right]+\frac{\mu}{\rho C_p}\left(\frac{\partial u}{\partial y}\right)^2, \tag{3}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D_B\frac{\partial^2 C}{\partial y^2}+\frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2}, \tag{4}$$

subject to the boundary conditions

$$u=u_w(x)=\lambda U_\infty,\ v=0,\ T=T_w,\ C=C_w \ \text{at}\ y=0,$$
$$u=U_\infty,\ T\to T_\infty,\ C\to C_\infty \ \text{as}\ \ y\to\infty, \tag{5}$$

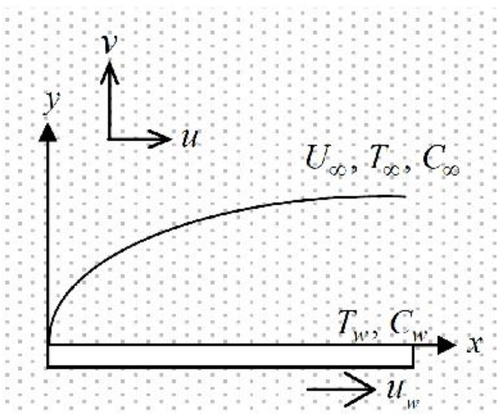

Fig. 1. Physical model of the coordinate system.

where u and v are the velocity components along the x and y directions, respectively, μ is the dynamic viscosity, ν is the kinematic viscosity, ρ is the fluid density, k is the thermal conductivity and C_p is the specific heat capacity at constant pressure. Next, D_B is the Brownian diffusion coefficient, D_T is the thermophoresis diffusion coefficient, τ is the ratio of the effective heat capacity of the nanoparticle material and the heat capacity of the ordinary fluid.

The set of similarity transformation for Eqs. (1) to (4) subjected to the boundary conditions (5) is (Weidman *et al.* 2006; Bachok *et al.* 2010)

$$\eta=\left(\frac{U_\infty}{2\nu x}\right)^{1/2}y,\ \ \psi=\left(2U_\infty\nu x\right)^{1/2}f(\eta),$$
$$\theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty},\ \phi(\eta)=\frac{C-C_\infty}{C_w-C_\infty}, \tag{6}$$

where ϕ and θ are the rescaled nanoparticle volume fraction and dimensionless temperature of the fluid, respectively. ψ is the stream function defined as $u=\frac{\partial\psi}{\partial y}$ and $v=-\frac{\partial\psi}{\partial x}$ which identically satisfy Eq. (1). Then, u and v can be derived as

$$u=U_\infty f'(\eta),$$
$$v=-\left(\frac{U_\infty\nu}{2x}\right)^{1/2}f(\eta)+\frac{U_\infty y}{2x}f'(\eta), \tag{7}$$

By substituting Eqs. (6) and (7) into Eqs. (2) to (4), we have

$$f'''+ff''=0 \tag{8}$$

$$\frac{1}{\Pr}\theta''+f\theta'+N_b\theta'\phi'+N_t\theta'^2+Ecf''^2=0, \tag{9}$$

$$\phi''+\frac{N_t}{N_b}\theta''+Lef\phi'=0 \tag{10}$$

Where $\Pr=\frac{\nu\rho C_p}{k}$ is the Prandtl number, $N_b=\frac{\tau D_B(C_w-C_\infty)}{\nu}$ is the Brownian motion parameter, $N_t=\frac{\tau D_T(T_w-T_\infty)}{T_\infty\nu}$ is the thermophoresis parameters, $Ec=\frac{(U_\infty)^2}{C_p(T_w-T_\infty)}$ is the Eckert number and $Le=\frac{\nu}{D_B}$ is the Lewis number.

The boundary conditions (5) become

$$f(0)=0,\ f'(0)=\lambda,\ \theta(0)=1,\ \phi(0)=1,$$
$$f'(\eta)\to 1,\ \theta(\eta)\to 0,\ \phi(\eta)\to 0,\ \text{as}\ y\to\infty \tag{11}$$

Note that $\lambda>0$ corresponds to downstream movement of the plate from the origin (Weidman *et al.* 2006). The physical quantities of interest are the skin friction coefficient C_f the local Nusselt number Nu_x and the local Sherwood number Sh_x

which are given by

$$C_f = \frac{\tau_w}{\rho u_e^2},\ Nu_x = \frac{x q_w}{k(T_w - T_\infty)},\quad Sh_x = \frac{x j_w}{D_B(C_w - C_\infty)}. \tag{12}$$

The surface shear stress τ_w, the surface heat flux q_w and the surface mass flux j_w are given by

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0},\ q_w = -k\left(\frac{\partial T}{\partial y}\right)_{y=0},\quad j_w = -D_B\left(\frac{\partial C}{\partial y}\right)_{y=0} \tag{13}$$

with $\mu = \rho\nu$ being the dynamic viscosity. Using the similarity variables in Eq. (6) give

$$C_f\left(2\mathrm{Re}_x\right)^{1/2} = f''(0),\ Nu_x\left(\frac{\mathrm{Re}_x}{2}\right)^{-1/2} = -\theta'(0),\quad Sh_x\left(\frac{\mathrm{Re}_x}{2}\right)^{-1/2} = -\phi'(0) \tag{14}$$

where $\mathrm{Re}_x = \frac{U_\infty x}{\nu}$ is the local Reynolds number. It is worth mentioning that the physical quantities of interest in the present context are

$$C_f\left(2\mathrm{Re}_x\right)^{1/2},\ Nu_x\left(\frac{\mathrm{Re}_x}{2}\right)^{-1/2} \text{ and } Sh_x\left(\frac{\mathrm{Re}_x}{2}\right)^{-1/2}$$

which are referred as the reduced skin friction coefficient, the reduced Nusselt number and the reduced Sherwood number and can be denoted as C_{fr}, Nur and Shr which are represented by $f''(0)$, $-\theta'(0)$ and $-\phi'(0)$, respectively.

3. NUMERICAL METHOD

The transformed ordinary differential equations subject to the associated boundary conditions are solved numerically using the Keller-box method. As described in the books by Na (1979) and Cebeci and Bradshaw (1988), the solution is obtained in the following four steps: Firstly, reduce Eqs. (8) to (11) to a first-order system. Secondly, write the difference equations using central differences; Thirdly, linearize the resulting algebraic equations by Newton‘s method, and write them in the matrix-vector form; and finally, solve the linear system by the block tridiagonal elimination technique.

4. RESULTS AND DISCUSSION

Six parameters are considered, namely the Prandtl number $\Pr$, the plate velocity parameter λ, the Brownian motion parameter N_b, the thermophoresis parameter N_t, the Eckert number Ec, and the Lewis number Le. The step size $\Delta\eta = 0.02$ and the boundary layer thickness η_∞ from 2.5 to 8 were used in obtaining the numerical results. From the literature review, in considering the boundary layer flow over a stretching surface in nanofluids, it was found that N_b and N_t effects are practically studied at range between 0.1 and 0.5 while Le in the range of 1 to 40 (Khan and Pop 2010; Noghrehabadi *et al.* 2012; Nandy and Mahapatra 2013). Therefore, in the present study, the same range is considered in our analysis and discussion.

Table 1 shows the comparison values of $-\theta'(0)/\sqrt{2}$ with previous results by Roşca and Pop (2014) for various values of Prandtl number $\Pr$. It is found that they are in good agreement. We can conclude that this method works efficiently for the present problem.

Table 2 shows the values of Nur, Shr and C_{fr} for various values of λ. From this table, it is found that the present of plate velocity parameter λ results to the increase of Nur and Shr. Meanwhile, the skin friction coefficient C_{fr} decreases as λ increases. Note that when $\lambda = 0$, $C_{fr} = 0.4696$ which is in a very good agreement with Blasius (1908). Next, $C_{fr} = 0$ for $\lambda = 1$ due to the plate and the fluid move with the same velocity which results in no velocity gradient, i.e. there is no friction at the fluid-solid interface.

Table 1 Comparison values of $-\theta'(0)/\sqrt{2}$ with previous published results for various values of Pr when $\lambda = N_b = N_t = Ec = Le = 0$

Pr	Roşca and Pop (2014)	Present
0.7	0.29268	0.292680
0.8	0.30691	0.306917
1	0.33205	0.332057
5	057668	0.576689
10	0.72814	0.728141

Table 2 Values of Nur, Shr and C_{fr} for various values of λ when $\Pr = 7$, $N_b = N_t = Ec = 0.1$ and $Le = 10$

λ	Nur	Shr	C_{fr}
0	0.3747	1.1672	0.4696
0.1	0.4705	1.3369	0.4625
0.5	0.7875	1.8657	0.3288
1	1.0717	2.3805	0
2	1.2994	3.3099	-1.0191

Table 3 presents the values of the reduced Nusselt number Nur and Sherwood number Shr for various values of N_b and N_t. From this table, it is concluded that the increase of both parameters N_b and N_t results to the decrease of Nur while Shr increases with the increase of N_t. Physically, it is suggested that the small values of N_b and N_t enhance convective heat transfer capabilities while the large values of N_b and N_t enhance the convective mass transfer capabilities.

Figures 2 to 4 illustrate the temperature profiles for various values of N_b, N_t, Le, Ec and ε. These Figs are plotted in order to understand the behavior of the thermal boundary layer thickness for various values of parameter discussed. From Fig. 2, it is found that as N_b increases, the thermal boundary layer thickness also increases. According to Anwar *et al.* (2013), this is due to the fact that larger values of N_b have a large extent of fluid hence thickening the thermal boundary layer thickness. The same trends happen for parameter N_t. This result from the deeper penetration into the fluid which cause to the thickening of thermal boundary layer thickness.

Table 3 Values of Nur and Shr for various values of N_b and N_t when $\Pr = 7$, $Ec = 0.1$, $Le = 10$ and $\varepsilon = 0.5$

N_b	N_t	Nur	Shr
0.1	0.1	0.7875	1.8657
0.2	0.1	0.4923	1.9951
0.3	0.1	0.2959	2.0091
0.4	0.1	0.1712	2.0010
0.5	0.1	0.0952	1.9884
0.1	0.2	0.6047	2.0601
0.1	0.3	0.4708	2.3624
0.1	0.4	0.3722	2.6994
0.1	0.5	0.2990	3.0319

In Fig. 3, it is found that the increase of Le results to the decrease in thermal boundary layer thickness. The opposite trend occurs for Ec where the increase of this parameter results to the increase of thermal boundary layer thickness. This may be explained as follows: The increase of Ec directly proportional to the increase of external velocity (see definition of Ec) and this situation reduces the effect of the plate velocity parameter ε, which thickening the thermal boundary layer thickness.

Figure 4 shows the temperature profiles for various values of ε. It is found that the present of plate velocity parameter $(\varepsilon > 0)$ reduces the thermal boundary layer thickness. Physically, the increase of ε results to the increase of ratio velocity differences between the plate and the fluid which enhance the fluid to move away from the plate region rapidly. This situation reduces the thermal diffusivity and thinning the thermal boundary layer thicknesses.

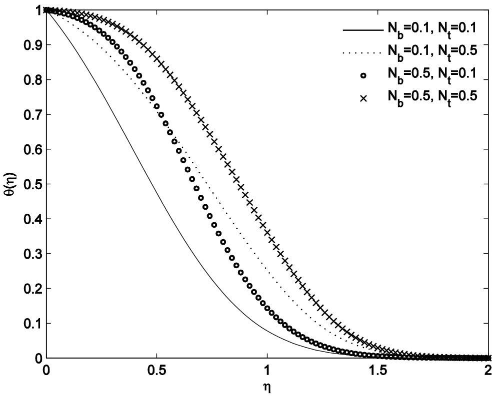

Fig. 2. Temperature profiles $\theta(\eta)$ for various values of N_b and N_t when $\Pr = 7$, $Ec = 0.1$, $Le = 10$ and $\varepsilon = 0.5$.

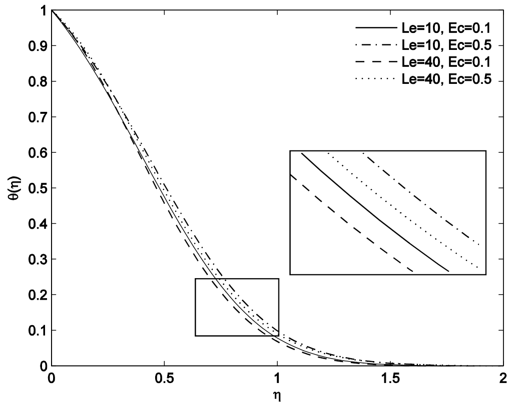

Fig. 3. Temperature profiles $\theta(\eta)$ for various values of Le and Ec when $N_b = N_t = 0.1$, $\Pr = 7$ and $\varepsilon = 0.5$.

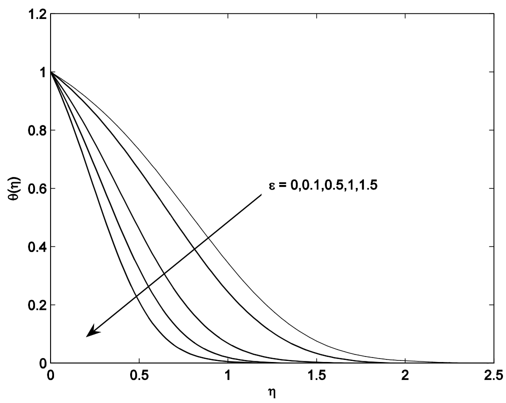

Fig. 4. Temperature profiles $\theta(\eta)$ for various values of ε when $N_b = N_t = 0.1$, $\Pr = 7$, $Le = 10$ and $Ec = 0.1$.

Figures 5 to 7 show the variation of nanoparticle volume fraction $\phi(\eta)$ for various values of parameters. The increase of N_b in Fig. 5 results to the decrease of $\phi(\eta)$. Also, the effects of ε is similar as in Fig. 4. The moving effects reduces the nanoparticle volume fraction $\phi(\eta)$.

The effects of N_t and Le are shown in Fig. 6. Both parameters consume the contra trend where the increase of Le is to the decrease the nanoparticle volume fraction $\phi(\eta)$ while $\phi(\eta)$ increases as N_t increases. Further, the nanoparticle volume fraction $\phi(\eta)$ increases monotonically as $\Pr$ decreases. Meanwhile, the Eckert number Ec shows the contradict trends. Large values of Ec means large viscous dissipation effect and this enhance the nanoparticle volume fraction $\phi(\eta)$. The variations of the nanoparticle volume fraction for both parameters $\Pr$ and Ec are plotted in Fig. 7.

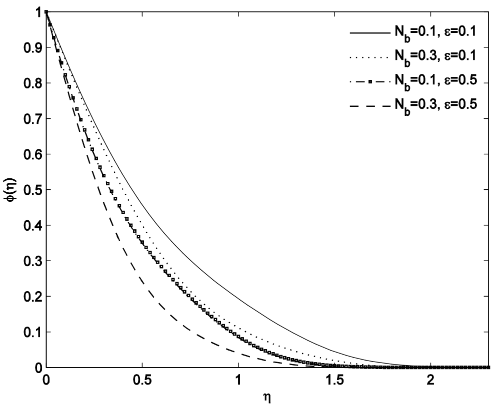

Fig. 5. Variation of nanoparticle volume fraction $\phi(\eta)$ for various values of N_b and ε when $\Pr = 7$, $N_t = Ec = 0.1$ and $Le = 10$.

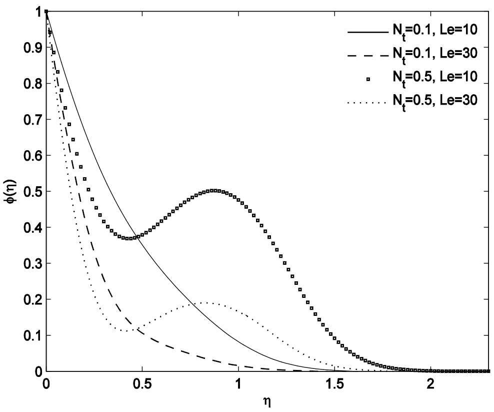

Fig. 6. Variation of nanoparticle volume fraction $\phi(\eta)$ for different values of N_t and Le when $\Pr = 7$, $N_b = Ec = 0.1$ and $\varepsilon = 0.5$.

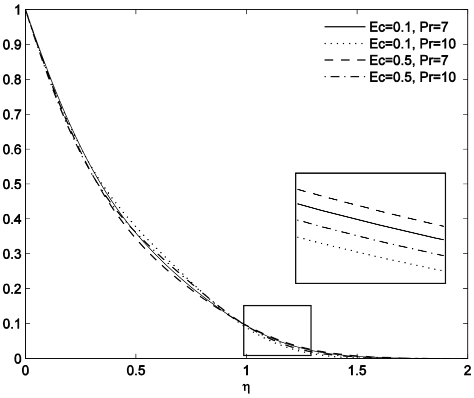

Fig. 7. Variation of nanoparticle volume fraction $\phi(\eta)$ for different values of $\Pr$ and Ec when $N_b = N_t = 0.1$, $Le = 10$ and $\varepsilon = 0.5$.

Figures 8 and 9 illustrate the variation of the reduced Nusselt number Nur and the reduced Sherwood number Shr with ε for various values of Ec. From Fig. 8, it is found that the presence of viscous dissipation $(Ec \neq 0)$ changes the variation of Nur curve to a quadratic curve. Also, the presence of Ec reduces the range of ε for which the solution exists It is clearly shown when $Ec = 0.1$, the physical acceptable solution occur until $\varepsilon = 4.0189$ while when the value of Ec increases $(Ec = 0.3)$, Nur stops at $\varepsilon = 2.7153$. Since $Nu \cong 0$, this means that there is no convection occur beyond this value, which leads to pure conduction heat transfer. Furthermore, it is noticed that when $\varepsilon = 1$, which result from the equivalent of stretching and external velocity at the plate surface, Ec does not effects the values of Nur.

In Fig. 9, it is noticed that the physical acceptable solution for ε has a similar trends as in Fig. 8. Further, it is found that, at large values of ε, larger values of Ec produce large values of Shr which physically means large in convective mass transfer. This is due to the increase of ratio of kinetic energy over enthalpy in increasing of Ec. Also, it is noticed that the variation of Shr is unique for all Ec when $\varepsilon = 1$ as in Fig. 8.

Figures 10 and 11 show the variation of Nur and Shr with $\Pr$ for various values of Le. From Fig. 10, it is found that small values of $\Pr$ such as liquid metal $(\Pr \ll 1)$, produce very small value of Nur which imply no convection occurs or the heat transfer is in pure conduction situation. It is realistic since liquid metals have high thermal conductivity but low in viscosity. For fixed values of $\Pr$, increasing Le results to the decrease of Nur but increase of Shr.

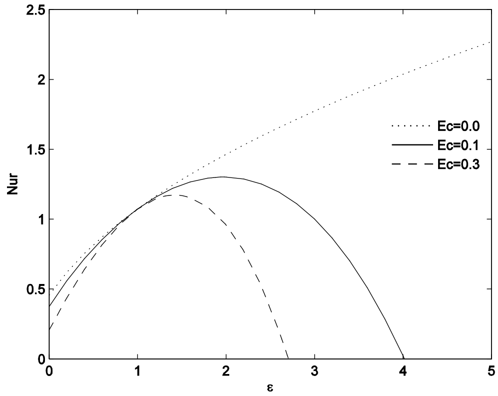

Fig. 8. Variation of reduced Nusselt number with ε for various values of Ec when $Pr = 7$, $Le = 10$ and $N_b = N_t = 0.1$.

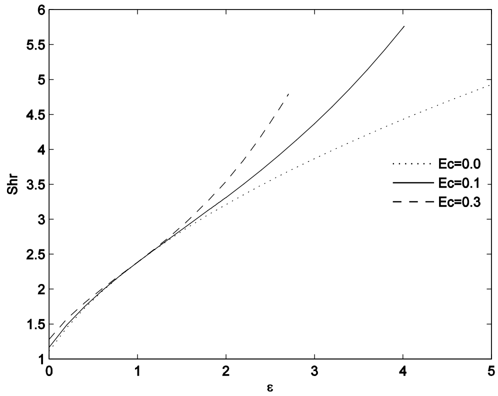

Fig. 9. Variation of reduced Sherwood number with ε for various values of Ec when $Pr = 7$, $Le = 10$ and $N_b = N_t = 0.1$.

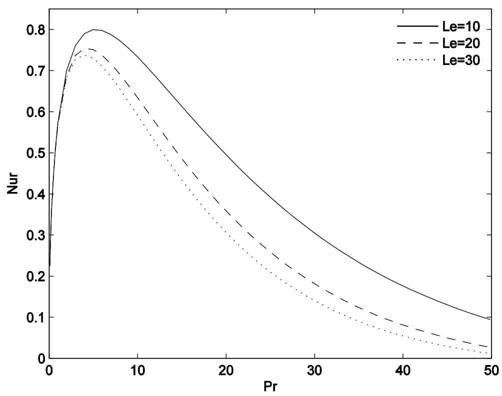

Fig. 10. Variation of reduced Nusselt number with Pr for various values of Le when $N_b = N_t = Ec = 0.1$ and $\varepsilon = 0.5$.

Lastly, Fig. 12 presents the variation of the reduced skin friction coefficient C_{fr} for various values of ε which produce $f'(0)=\varepsilon$ and $f'(\eta)=1$ as $\eta \to \infty$. From this figure, it is found that the value of C_{fr} is positive for $\varepsilon<1$, and C_{fr} decreases to 0 as ε approach 1. The value $C_{fr}=0$ at $\varepsilon=1$ is due to the plate moves in the same velocity with the fluid. Further, from this figure, we can conclude that C_{fr} decreases as ε increases.

From this study it is worth mentioning that only moving parameter affects the velocity profiles and the value of the skin friction coefficient. It is clear from the ordinary differential Eqs. (8) to (10) and boundary conditions (11).

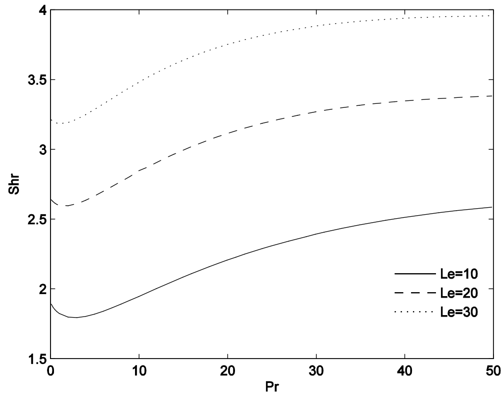

Fig. 11.Variation of reduced Sherwood number with Pr for various values of Le when $N_b = N_t = Ec = 0.1$ and $\varepsilon = 0.5$.

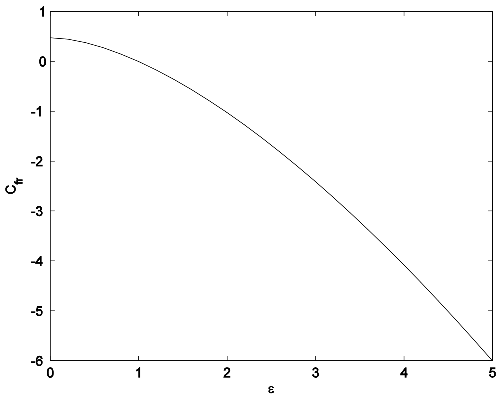

Fig. 12. Reduced skin friction coefficient for various values of ε.

5. CONCLUSION

In this paper, the boundary layer flow over a moving plate in a nanofluid in the presence of viscous dissipation was numerically studied. It was shown how the Prandtl number Pr, plate velocity parameter ε, Brownian motion parameter N_b, thermophoresis parameter N_t, Eckert number Ec and Lewis number Le affect the values of the reduced Nusselt and Sherwood numbers and the skin friction coefficient as well as the concentration and temperature profiles. As a conclusion, the thermal boundary layer thickness depends strongly on these parameters. It was found that in the presence of viscous dissipation, the range of the plate velocity parameter for which the solution exists reduces, which physically leads to pure conduction to occur. Further, the same situation occurs to the nanofluid with low Prandtl number. This may be explained as the low Prandtl value means the fluid is low in viscosity but high in thermal conductivity such as liquid metal. The increase of N_b, N_t and Ec results to the increase in thermal boundary layer thickness. Meanwhile, the increase of $\Pr$, £ and Le reduced the thermal boundary layer thickness.

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