On the Use of the Water Hammer Equations with Time Dependent Friction during a Valve Closure, for Discharge Estimation

G. Dunca[1], R. G. Iovănel[1], D. M. Bucur[1†], and M. J. Cervantes[2,3]

[1] University POLITEHNICA of Bucharest, Romania
[2] Luleå University of Technology, Luleå, Sweden
[3] Norwegian University of Science and Technology, Trondheim, Norway

†*Corresponding Author Email: dmbucur@yahoo.com*

ABSTRACT

The paper presents a new method for in site discharge estimation in pressured pipes. The method consists in using the water hammer equations solved with the method of characteristics with an unsteady friction factor model. The differential pressure head variation measured during a complete valve closure is used to derive the initial flow rate, similarly to the pressure-time (Gibson) method. The method is validated with a numerical experiment, and tested with experimental laboratory measurements. The results show that the proposed method can reduce the discharge estimation error by 0.6% compared to the standard pressure-time (Gibson) method for the flow rate investigation.

Keywords: Pressure measurement; Method of characteristics; Unsteady friction factor; Discharge evaluation; Hydropower.

NOMENCLATURE

A	cross-section area of the pipeline	q	leakage flow
a	pressure wave speed	Q	steady-state discharge
B_0	pressure amplitude corresponding to the fundamental harmonic of the free pressure oscillation	Q_0	initial guess for discharge
		Q_{ref}	reference value for discharge
		T	the pressure wave period
C^-	negative characteristic	t	independent temporal variable
C^+	positive characteristic	t_1	beginning of the analysed time-history
C^*	vardy's shear decay coefficient	t_2	end of the initial steady state
D	pipe diameter	t_3	end of the transient state corresponding to the forced flow rate change
dH	pressure head difference		
dH_0	steady state pressure head difference	t_4	end of the analysed time-history
E	pipe walls Young modulus	V	bulk velocity
e	bulk modulus	x	independent spatial variable
f	friction factor		
f_q	quasi-steady friction factor	Δp	differential pressure
g	acceleration due to gravity	ε	pipe wall thickness
H	pressure head	μ	arbitrary value
h	the oscillation damping decrement	ω	the circumferential wave frequency
k	brunone friction coefficient	ζ	pressure drop due to viscous losses
L	length of the measuring segment	ρ	density of the flowing liquid

1. INTRODUCTION

In the field of efficiency measurement with application to hydraulic machinery, the discharge is the most difficult parameter to determine accurately. The more precise is its estimation, the more reliable the turbine hydraulic efficiency will be.

For in site discharge measurement, there are several methods presented in the international standard IEC 60041 (1991) which can be applied for discharge measurement in pipes: velocity-area method, pressure-time or tracer methods. The thermodynamic and ultrasound methods may also be used. All of these methods have a good accuracy if used in the recommended conditions. The pressure-time method is the simplest in terms of costs and requirements when pressure taps are available.

The pressure-time method, Gibson (1923), is based on the second law of Newton, conservation of momentum, and consists in measuring the pressure difference between two pipe cross-sections, during a flow stop due to a valve closure for example. The discharge is computed using the following equation:

$$Q = \frac{A}{\rho L} \int_0^t (\Delta p + \zeta) dt + q ,\qquad (1)$$

where: Q is the unknown discharge before closure, A is the pipe cross-section area, ρ is the liquid density, Δp is the measured pressure difference, ζ is the pressure loss between the two measuring cross-sections, L is the length between the two measuring sections and q is the leakage flow after the valve closure, present when some gaps exist after complete closure.

The method requires certain conditions for a good flow estimation: the distance between the two measuring cross-sections (the measuring length) must be greater than 10 m ($L > 10$ m), the product between this distance and the steady state flow velocity must be higher than 50 m²/s ($V \cdot L > 50$ m²/s).

The improvement of the discharge determination method has concerned the researchers in the last years. Jonsson *et al.* (2012) developed the unsteady Gibson method by implementing an unsteady friction factor, using the Brunone model. The obtained procedure was tested and validated in situations outside the standard limitations, using numerical and laboratory data. Later, the method was validated with on-site experimental tests by Dunca *et al.* (2013).

Adamkowski and Janicki (2010), Adamkowski (2012) showed that the standard pressure-time equation for discharge estimation doesn't consider the residual pressure oscillations that appear after the valve closure. In order to accomplish that, he introduced a term that modifies the integration upper limit for the estimation of the discharge in Eq. (1). The results obtained by applying this procedure were shifted systematically towards a lower value of the discharge. This led to a shift in the discharge estimation error, but did not increase the estimation precision in all analysed cases.

Further, Adamkowski and Janicki (2013) developed another procedure that considers both the liquid compressibility and the pipe walls deformability via the speed of sound, *a*. It uses the water hammer equations in which the quantity represented by the

pressure head H was replaced by the pressure head difference dH. The method of characteristic (MOC) was used to solve the equations.

In the present work, a development based on the method described by Adamkowski and Janicki (2013) using the water hammer equation is proposed, by using an unsteady model for the friction factor instead of a constant one. The numerical implementation of the model is made considering the most appropriate numerical scheme (explicit derivative scheme or implicit derivative scheme). The resulted procedure is tested on a numerical set of data and validated with experimental data. The method accuracy is compared to those achieved with the standard pressure-time and unsteady pressure-time methods.

2. METHOD

The equations describing the fast variation of the flow in 1-dimensional pressurized pipes are the continuity equation and the momentum equation (Eq. (2) and Eq. (3)).

$$F_1 = \frac{a^2}{g} \frac{\partial V}{\partial x} + \frac{\partial H}{\partial t} + V \frac{\partial H}{\partial x} = 0 \qquad (2)$$

$$F_2 = \frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} + g \frac{\partial H}{\partial x} + f \frac{V|V|}{2D} = 0 \qquad (3)$$

where: x is the axial distance, t is the time, V is the mean flow velocity, H is the pressure head, g is the acceleration due to gravity, f is the friction factor, D is the pipe diameter and a is the pressure wave speed. The parameter a depends on the pipe walls Young modulus E, liquid density ρ, and bulk modulus e, pipe wall thickness, ε and diameter, D, according to the relation: $a = \sqrt{\varepsilon/\rho} / \sqrt{1 + (\varepsilon D)/(eE)}$.

These equations can be solved using various numerical methods (Karney and McInnis, 1992, Chaudlhry, 1987). The method of characteristics (MOC) is one of the most frequently used methods to solve this system. It consists in writing the two equations as a linear combination of them, as:

$$F = F_1 + \mu F_2 = 0 \qquad (4)$$

where the parameter μ has an arbitrary value. Thus, Eq. (4) becomes:

$$\left[\frac{\partial H}{\partial t} + (V + \mu g) \frac{\partial H}{\partial x} \right] +$$
$$\mu \left[\frac{\partial V}{\partial t} + \left(V + \frac{a^2}{\mu g} \right) \frac{\partial V}{\partial x} \right] + \mu f \frac{V|V|}{2D} = 0 \qquad (5)$$

where two values for the parameter μ are selected to obtain the total derivatives of H and V between the brackets. These are $\mu = \pm a/g$ and then, according to Wylie and Streeter (1993), Eqs. (2) and (3) can be replaced by two ordinary differential equations, i.e., Eq. (6) valid along the positive characteristic, C$^+$ ($dx/dt = a$), and Eq. (7) valid for the negative

characteristic, C⁻ ($dx/dt = -a$), respectively:

$$\frac{dH}{dt} + \frac{a}{g}\frac{dV}{dt} + af\frac{V|V|}{2gD} = 0 \tag{6}$$

$$\frac{dH}{dt} - \frac{a}{g}\frac{dV}{dt} - af\frac{V|V|}{2gD} = 0 \tag{7}$$

In this way, a relation between the flow parameters, V and H, during the water hammer phenomenon is determined. To solve Eqs. (6) and (7), an explicit numerical scheme is employed with diamond grid scheme for interpolation, as recommended by Vitkovsky *et al.* (2000). Because $V \ll a$, on a grid as in Fig. 1, the following discrete expressions are obtained along C⁺ ($dx/dt = a$) from Eq. (6), and along C⁻ ($dx/dt = -a$) from Eq. (7), respectively according to Bergant *et al.* (2001):

$$H_P - H_M + \frac{a}{g}\left(V_P - V_M\right) + \frac{f \cdot \Delta x}{2g \cdot D} \cdot V_P \cdot |V_M| = 0 \tag{8}$$

$$H_P - H_N - \frac{a}{g}\left(V_P - V_N\right) - \frac{f \cdot \Delta x}{2g \cdot D} \cdot V_P \cdot |V_N| = 0 \tag{9}$$
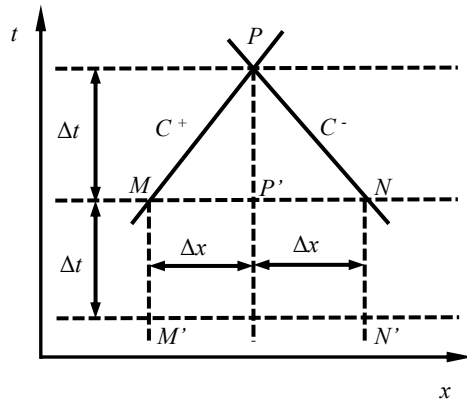

Fig. 1. Characteristics in the plane *xOt*.

By solving the two equations, the flow parameters V and H in the current point P can be obtained.

The model proposed in the present paper is based on rewriting the water hammer classical equations in the form presented by Adamkowski and Janicki (2013). The pressure head H and the flow velocity V are replaced with the pressure head difference, dH, between two cross-sections and the discharge, Q (Eq. 10 and Eq. 11):

-   along the positive characteristic C⁺

$$dH_P - dH_M +$$

$$+\frac{a}{g \cdot A}\left(Q_P - Q_M\right) + \frac{f \cdot \Delta x}{2g \cdot D \cdot A^2} \cdot Q_P \cdot |Q_M| = 0 \tag{10}$$

-   along the negative characteristic C⁻

$$dH_P - dH_N -$$

$$-\frac{a}{g \cdot A}\left(Q_P - Q_N\right) - \frac{f \cdot \Delta x}{2g \cdot D \cdot A^2} \cdot Q_P \cdot |Q_N| = 0 \tag{11}$$

The model developed by Adamkowski and Janicki (2013) aims to obtain the discharge flowing through a pipe using the pressure head difference measured between two cross-sections, as the pressure-time method. In the method, the effects of liquid compressibility and pipe walls deformability are considered using the Eq. (10) and (11) via the speed of sound, a. The computational procedure implies defining certain moments in time, which characterize the water hammer transient phenomenon (Fig. 2):

-   t_1 – beginning of the analysed time-history

-   t_2 – end of the initial steady state

-   t_3 – end of the transient state corresponding to the forced flow rate change

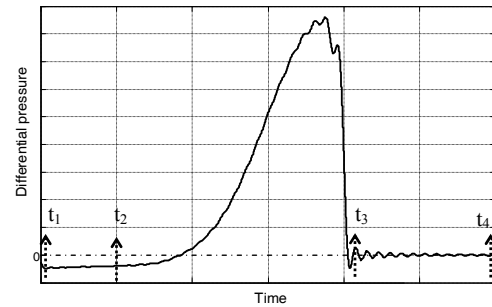- t_4 – end of the analysed time-history.

Fig. 2. Differential pressure variation and time definition.

The values of t_2 and t_3 correspond to the time interval in which the flow is completely stopped. The time corresponding to the end of the transient stage, t_3, is difficult to define being a present research subject. In the classical pressure time method for discharge determination, the IEC 41 standard presents a way to estimate this final integration time. Adamkowski (2012) proposed a way to determine this integration time by solving the definite integral:

$$\int_0^\tau B_0 e^{-ht} \cos(\omega t) dt =$$

$$= \frac{B_0}{h^2 + \omega^2}\left\{e^{-h\tau}\left[-h\cos(\omega\tau) + \omega\sin(\omega\tau)\right] + h\right\} = 0 \tag{12}$$

where B_0, $\omega = 2\pi/T$, $h = (1/T) \cdot \ln(B_i/B_{i+1})$ and T are the pressure amplitude corresponding to the fundamental harmonic of the free pressure oscillation, the circumferential wave frequency, the oscillation damping decrement and the pressure wave period (Fig. 3). The solution of Eq. (12) is the value of the time corresponding to the end of the transient stage, $\tau = t_3$.
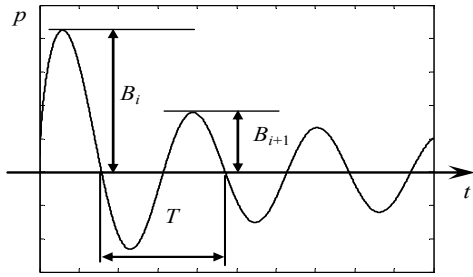
Fig. 3. Free pressure oscillation after valve closure.

In the method presented by Adamkovski, the friction factor f is considered constant. This hypothesis is acceptable for pipes with high roughness and a quasi-steady-transient phenomenon, i.e., slow transient. For unsteady flows, there are several friction models presented in the literature (Karney and McInnis, 1992, Bergant *et al.*, 2001, Bahrani and Nouar, 2014). Bergant *et al.* (2001) analysed some of the friction models obtaining the best results with the Brunone model. This model gave good results in other studies as Jonsson *et al.* (2012) and Dunca *et al.* (2013). In the present work, the model is implemented in the method proposed by Adamkowski to evaluate possible improvement in the error associated with the discharge estimation.

The Brunone model is described by Bergant *et al.* (2001). It consists in expressing the friction factor f as:

$$f = f_q + \frac{k \cdot D}{V \cdot |V|} \left(\frac{\partial V}{\partial t} - a \cdot \frac{\partial V}{\partial x} \right) \tag{13}$$

where f_q is the quasi-steady friction factor, k is the Brunone friction coefficient, $\partial V / \partial t$ is the instantaneous local acceleration and $\partial V / \partial x$ is the instantaneous convective acceleration. The coefficient k can be determined either by trial and error method or analytically using the Vardy's coefficient (Vardy's shear decay coefficient C^*), $k = \sqrt{C^*}/2$, empirically calibrated. Coefficient C^* is 0.00476 for laminar flows while for turbulent flows is computed using the equation:

$$C^* = \frac{7.41}{Re^{\log\left(14.3/Re^{0.05}\right)}} \tag{14}$$

The quasi-steady part of the friction factor, f_q, is computed using Darcy equation for laminar flow ($f_q = 64/Re$) and the Haaland equation for turbulent flow:

$$\frac{1}{\sqrt{f_q}} = -1.8 \log \left[\left(\frac{\varepsilon/D}{3.7} \right)^{1.11} + \frac{6.9}{Re} \right] \tag{15}$$

The Brunone model was numerically implemented by solving the time derivative (local instantaneous acceleration) and the space derivative (convective instantaneous acceleration) with a diamond grid in

an explicit derivative scheme Vitkovsky *et al.* (2000).

Using the notations from Fig. 1, the local instantaneous accelerations from Eq. 13 are evaluated by $(\partial V/\partial t)_M = (V_M - V_{M'})/\Delta t$ and $(\partial V/\partial t)_N = (V_N - V_{N'})/\Delta t$, while the convective accelerations by $(\partial V/\partial x)_M = (V_{P'} - V_M)/\Delta x$ and $(\partial V/\partial x)_N = (V_N - V_{P'})/\Delta x$.

In order to evaluate the flow rate with the proposed evaluation procedure, the following information is needed:

- pressure head difference, dH, measured between two cross-sections. An initial discharge value is imposed as initial guess for this code.

- definition of the moments t_1, t_2, t_3 and t_4 based on the pressure head difference dH. The time value t_3 is determined by solving the Eq. (12).

- geometrical characteristics of the pipe (D – diameter, E – pipe walls Young modulus, L – distance between the pressure head measuring sections), and the liquid properties (ρ – density, e – bulk modulus).

A discretization grid is generated in the space-time domain, xOt (Fig. 1), in order to apply the method of characteristics. The procedure is iterative. An initial guess for the steady state discharge, Q_0, is firstly made, and a value for f before valve closure is obtained by

$$f = (2 \cdot g \cdot D \cdot dH_0) \Big/ \left[L \left(\frac{Q_0}{A} \right)^2 \right] \tag{16}$$

where dH_0 is the measured pressure difference in the steady state regime.

Starting with these values for Q_0, and f, in all grid points at the time t_1, the MOC is applied during $t_1 - t_4$ time-interval, using the boundary conditions:

- at upstream end (first measuring section):

$dH(t) = 0$, while $Q(t)$ results from Eq. (11) along C⁻

- at downstream end (second measuring section):

$dH(t)$ according with measured data, while $Q(t)$ results from Eq. (10) along C⁺

For any grid points in-between, the Eqs. (10) and (11) are used, to obtain the time variations of the pressure head difference dH and the discharge Q.

In all computations, the Brunone unsteady friction factor, Eq. (13), is taken into account by the presented explicit scheme.

A new value of the steady state flow rate Q_0, is then derived as the average value of the discharge trace during the steady state $t_1 - t_2$ time-period.

The obtained discharge value is compared with the previous one and if the difference between them is less than an imposed value the computation stops. If this condition is not accomplished, the computation

resumes with the new Q_0.

3. NUMERICAL VALIDATION

The proposed model was developed in the MATLAB software and validated using a numerically generated case from which time variation of pressure head difference during a valve closure were obtained.

In order to test the efficiency of the proposed method outside the limitations stipulated in the IEC 41 standard for the pressure-time method, the time variations of the pressure head are numerically created considering two cases with a distance of 0.9and 9 m between the two measuring sections. The data were generated considering the Brunone unsteady friction factor model and a valve closing function derived from experiments made by Jonsson *et al.* (2012). The normalized valve closure with time duration of 4 s is presented dimensionless in Fig. 4 and named here "experimental closure".
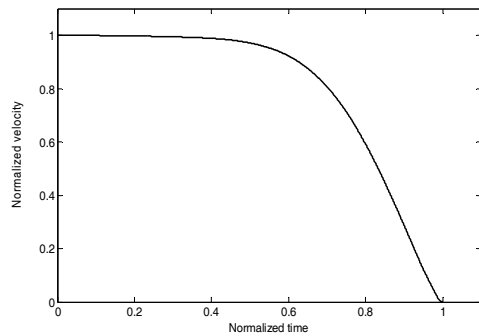
Fig. 4. Valve closure function.

Case 1 is a 4 m long pipe, with an inner diameter D = 0.3 m. The water is supplied from a tank with a 33.53 m head. The differential pressure head were obtained by solving the water hammer equations with MOC in two cross-sections located at 0.9 m one from the other. The two measuring cross-sections are located at 0.4 and 1.3 m from the downstream valve, respectively.

Case 2 is a pipe with a length of 40 m, an inner diameter D = 0.3 m. The water is supplied from a tank with 33.53 m head. The considered upstream and downstream sections were located at 9 m one from the other. The two measuring cross-sections are located at 4 and 13 m from the downstream valve, respectively.

In both cases, the pressure wave speed was considered equal to 900 m/s.

Three values of discharge 0.16, 0.3 and 0.4 m³/s were considered to obtain pressure head variations. For both values of length and the three discharge values, 4 different methods for discharge evaluation were applied:

- standard pressure-time method– steady Gibson

- the method developed by Jonsson *et al.* (2012) – unsteady Gibson

- the method developed by Adamkowski and Janicki (2013) – steady Adamkowski

- the proposed method – here named *unsteady Adamkowski*

The results are presented in figures 5 and 6 as relative errors of discharge obtained with each method compared to the reference value Q_{ref}, for the numerical generated data.

$$\varepsilon[\%] = \left(Q - Q_{ref}\right) / Q_{ref} \cdot 100 \qquad (17)$$

Figure 5 shows that regardless of the distance between the measuring sections for pressure head difference the methods steady Gibson and steady Adamkowski provide approximately equal results: 0.5% error for the discharge value 0.160 m³/s, -0.05% for 0.3 m³/s and -0.2% for 0.4 m³/s.
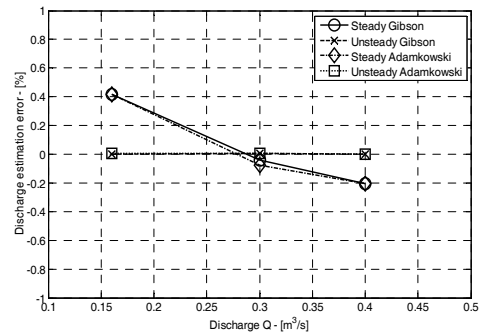

Fig. 5. Discharge evaluation error for Case 1 - measuring length of 0.9 m.

Both unsteady evaluation procedures (unsteady Gibson and unsteady Adamkowski) gave errors close to zero, as the differential pressure head data were obtained using the same unsteady friction model. In this way, the methods were in fact validated. With the two unsteady evaluation procedures, the errors were of 0.006% for the discharge value 0.160 m³/s, 0.003% for 0.3 m³/s and 0.002% for 0.4 m³/s. The difference between the steady and unsteady methods points out the potential error induced by assuming a constant friction factor.

In Fig. 6, the steady evaluation procedures provide also similar results. Some differences appear between the two unsteady evaluation procedures results. The two unsteady methods are similar, beside the liquid compressibility and pipe walls deformability effects taken into account in the proposed method. A sensibility analyse was performed to highlight this difference.

A new simulation was performed in order to obtain a differential pressure head trace with a sharper closure of the valve (Fig. 7). The results from the four evaluating procedures using the new generated data are presented in Table 1.
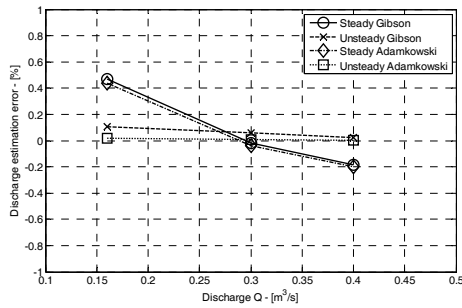
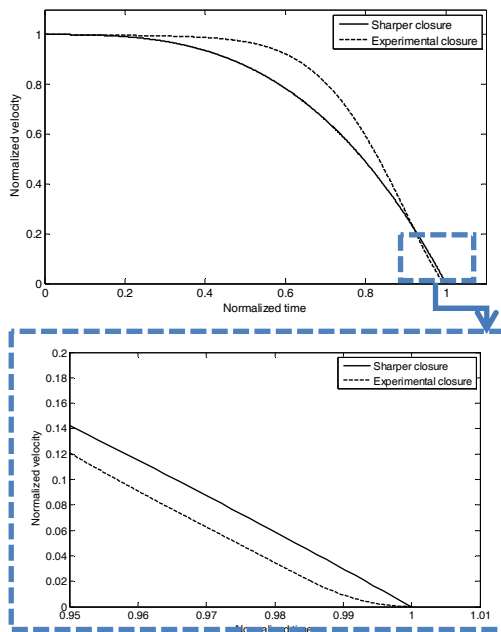**Fig. 6. Discharge evaluation error for Case 2 - measuring length of 9 m.**

Fig. 7. Sharper valve closure function.

A sharper valve closure leads to an increase of the difference between the unsteady methods. This emphasises the influence of liquid compressibility and pipe walls deformability over the method precision.

Table 1 Discharge estimation errors for new simulated data with sharper closure

Method	Error [%]			
	Steady Gibson	Unsteady Gibson	Steady Adamkowski	Unsteady Adamkowski
Sharper closure	0.38	0.18	0.38	0.05
Experimental closure	0.47	0.14	0.47	0.06

Another important aspect of the method is the correct choice of the time when the free pressure oscillations begin to occur, t_3. In his study, Adamkowski and Janicki (2013) emphasized that his method is very sensitive to it. Different values for t_3 were tested with Adamkowski method in Case 1 and the discharge estimation error was not significantly influenced. For Case 2, the correct choice of t_3 had an important effect over the error by doubling it. The time t_3 is thus of importance when the compressibility effects becomes significant.

4. EXPERIMENTAL VALIDATION OF THE MODEL

The second step of the study was testing the proposed method using experimental data. The available pressure traces were measured at the Waterpower Laboratory at NTNU (Jonsson, 2011) for three discharge values: 0.160, 0.300 and 0.400 m³/s. The test rig (Fig. 8) consisted in a pipeline system supplied from a tank. The hydraulic head of the system was 9.75 m. The testing section of the setup had a length of 26.67 m and an inner diameter of 0.3 m. The pressure wave speed was determined from the experiment measurements to 900 m/s. The measuring sections for the differential pressure transducer were located at 9 m one from another. The first section was 4 m from the valve. The differential pressure transducer has a range of ±0.5 bar and the accuracy of 0.25%. A magnetic flowmeter with an accuracy of 0.1% was mounted on the rig, so the discharge values measured could be used as reference for the accuracy analysis of proposed method. For each of the three discharge values, several tests were made, so the repeatability of the measurements could be assessed.
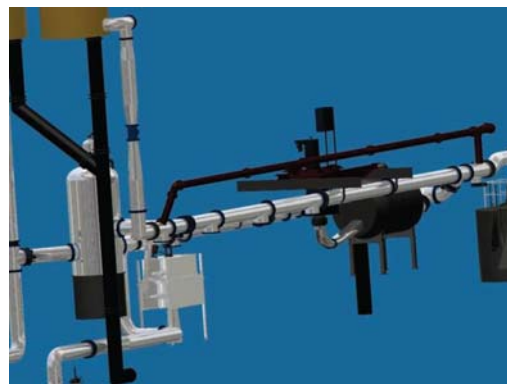

Fig. 8. CAD drawing of the test rig (Jonsson, 2011).

The discharge was estimated using the four procedures analysed in Section 3 with the numerical experiment. The results are presented as values of the relative error of the obtained discharge compared to the reference values in Fig. 9.

The steady Adamkowski method gives the same result as the standard Gibson method for the lower discharge value (Q = 0.160 m³/s), but for the other two discharge values, the first method is less precise. Comparing the unsteady Gibson method and the unsteady Adamkowski method, the second one has smaller errors in discharge evaluation.

The error values of the proposed method (unsteady Adamkowski) do not exceed ±0.1% for the range of investigated discharge values. In case of the lower

value of discharge, 0.160 m³/s, implementing the unsteady friction factor in Adamkowski method leads to a decrease of the discharge evaluation error by almost 0.6%. The proposed method gives better results than the standard Gibson and the unsteady Gibson methods, the discharge estimation error being decreased with 0.35% compared with the unsteady Gibson method error. This is due to the importance of liquid compressibility and pipe walls elasticity, which are not considered in the two mentioned methods.
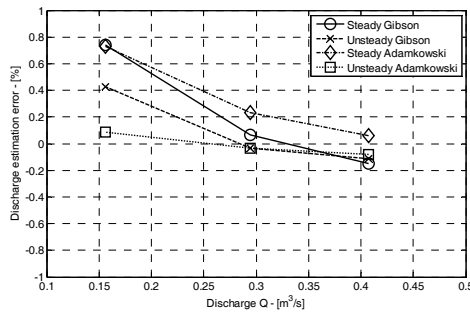

Fig. 9. Discharge evaluation error using laboratory data.

For the other two discharge values the errors obtained using the proposed method are comparable with those obtained with the unsteady Gibson method: 0.03% for 0.300 m³/s and -0.08% for 0.400 m³/s.

The influence of the correct choice of the moment when the free pressure oscillations occur, t_3, was considerable in this phase of the study. Choosing another moment then the one indicated by Adamkowski and Janicki (2013) in his study doubles the discharge evaluation error.

5. CONCLUSIONS

The paper presents a new model developed to evaluate the discharge based on pressure head time variation measurements during transient regimes, similar to the standard pressure-time method (also known as Gibson method).

The model consists in solving the water hammer equations in which an unsteady friction factor model is implemented. As boundary condition, the differential pressure head measured between two sections, during the entire transient regime is used. The novelty of the method resides into taking into account both the liquid compressibility and the pipe walls deformability, and the unsteady character of the hydraulic losses, unlike the previous methods.

The model was validated in a numerical experiment and used to evaluate the discharge based on laboratory measurements. In the numerical experiment two lengths of the pipe were considered (4 and 40 m) and the differential pressure head was extracted between two sections situated at 0.9 and 9 m one from the other (both measuring lengths being

lower than the standard limit of 10 m). The results showed that the effect of liquid compressibility and pipe walls deformability influences the discharge evaluation procedures accuracy. Also, taking into account the correct time for the end of the transient stage, the error can be considerably decreased.

Using the laboratory experimental data, the proposed method obtained a discharge estimation error between ±0.08% for the entire analysed discharge values range. Implementing the unsteady friction factor together with the correct choice of the end of the transient stage led to an estimation error reduction of about 0.6%, which can be very important in site efficiency tests.

In future works, the proposed method for discharge evaluation will be used on in site measured detain order to test its efficiency for different discharge value ranges and in different evaluation conditions.

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