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Existence of Subsonic Flow in Divergent Section Adjacent to Throat of a Convergent-Divergent Nozzle for Actual Flow

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ABSTRACT

In this study, a mathematical analysis by considering the effect of an actual index of expansion clearly shows a persistent of the existence of subsonic flow after the throat to a down stream in the region of divergent part to produce a supersonic velocity at the exit of a convergent-divergent nozzle. The length of the divergent part where subsonic velocity found is dependent upon the magnitude of the nozzle efficiency and the actual index of expansion. The change in velocity from subsonic, sonic and supersonic occurs only in the divergent part while the corresponding frictionless behavior has the classical features (subsonic in the convergent, sonic at the throat, and supersonic in the divergent). This is mainly due to thermodynamic processes which result a change in enthalpy due to friction and a gain in entropy. The reference conditions are newly derived for an actual frictional flow condition. This design aspect differs in a physical manner corresponding to that from an isentropic flow. An actual nozzle shape for convergent-divergent nozzles is also investigated under a non-isentropic flow condition.

Keywords: Actual shape; Convergent-divergent nozzle; Friction; Reference condition; Subsonic velocity.

NOMENCLATURE

A	cross section area	N	number of small stages of expansion
C	flow velocity	p	fluid pressure
c_1, c_2, c_3	integration constants	q	heat transfer rate
h	enthalpy	u	specific internal energy
h_a	enthalpy at inlet in a stage	v	specific volume
h_b	enthalpy at exit in a stage expanded isentropically	γ	isentropic index of expansion
$h_{b'}$	actual enthalpy at exit in a stage	η_n	nozzle efficiency
M	Mach number		
m	actual adiabatic index of expansion		

Subscript

MBED Equation.DSMT4 0 stagnation properties

1. INTRODUCTION

Nozzles and diffusers are energy transforming fluid passages that are meant for accelerating and decelerating fluid, respectively. These are extensively used in turbo machines and other types of machines. In steam turbines, nozzles are used to produce a high velocity steam. When steam flows through a nozzle, it expands from a high pressure to a low pressure, and as a result, steam velocity and specific volume increase well. In a nozzle with varying cross sectional area, pressure energy of steam is converted into kinetic energy in the flow direction with a high efficiency. For this conversion, different shapes of nozzles are used depending upon the basic requirement of the design condition.

The main goal of using a convergent-divergent nozzle is to achieve a supersonic velocity of fluid at the exit from a subsonic velocity. In a convergent-divergent nozzle, the flow velocity for an ideal frictionless condition reaches to a sonic velocity at the region of the minimum cross section area known as throat – this happens if the throat pressure to inlet pressure ratio is equal to the critical pressure ratio. Further, a decease in this nozzle pressure ratio which doesn't increase the mass flow rate called the choked condition. The flow after the throat section is free to expand to have a supersonic velocity in the divergent part. However, flow velocity of an actual situation inside a nozzle always differs from that of the ideal condition.

Bayt (1999) performed the modeling, design and testing of micro scale convergent-divergent nozzles. The flow in the nozzle was modeled by two dimensional finite volume Navier-Stokes simulations. Yuan (2008) provided an approximate model for the solution of the full Euler system with considering compressible flows in a Riemannian manifold and studied the stability of subsonic flows. Majdalani and Maicke (2013) calculated average local Mach number in convergent-divergent nozzles. In general, for flow through nozzles, hydrodynamic boundary layer is developed. The effect of boundary layer on choked flow in a convergent nozzle had been studied by Kubo *et al.* (2010). For the purpose being the generation of mechanical power, Vahaji *et al.* (2014) studied experimentally the performance of a two-phase nozzle as an expander. Their experiments involved water having temperature lower than 100^0C through a convergent-divergent nozzle to a low pressure flash tank for the thermal energy of steam converted into the kinetic energy. Using the measure data, the efficiency of the conversion process was evaluated. In order to obtain a higher cooling capacity in refrigeration cycles, throttle valve was replaced by the ejector (Wongwises and Disawas, 2005). The influence of the outlet diameter of the motive nozzle in the refrigeration process was investigated by Chaiwongsa and Wongwises (2008). For humid air flowing through nozzles, Dykas *et al.* (2017) investigated numerically the thermal and flow phenomena.

The classical analysis of convergent-divergent nozzles based on the isentropic process has demonstrated the sonic velocity at the throat section and supersonic velocity in the divergent section (Fox *et al.*, 2004). However, for an actual flow, friction always presents in the flow process and the isentropic condition doesn't satisfy ever. Therefore, it is noted that with consideration of the friction phenomenon, the classical nozzle analysis is required to modify by introducing the nozzle efficiency. Adopting suitable design variables, the actual analysis can be carried out.

Many researchers had studied the reason for the actual existence of the subsonic flow well after the throat section in the divergent part in case of frictionally resisted expansion. Xu and Zhao (2007) investigated the effect of shock waves and boundary layer separation for fluid flowing through a convergent-divergent nozzle. They observed an existence of subsonic zone in flow region at the throat area, which primarily due to a reduction of effective area ratio for the development of the viscous boundary layer. This effect causes the sonic velocity not to be at the throat location but a little bit downstream from the throat. Tsuge (2015) analyzed the adiabatic gas flow in a convergent-divergent nozzle with friction. He/She determined mathematically the effect of friction on flow processes, and established an approximate solution by using the Godunov scheme and fractional step procedure. He/She also demonstrated that the frictional effect is responsible for obtaining the sonic state to a down stream from the throat section.

A challenging problem in aerodynamics is to analyze an actual flow field in de-Laval nozzles (convergent-divergent type) for a prescribed pressure at its entrance and exit. Most of the researchers studied the various flow stages of fluid flow in the convergent-divergent nozzle under an isentropic expansion, but a very few established a mathematical solution while incorporating the friction and its effect on flow pattern in nozzles. From the design point of view, fluid flow with friction is always present. However, no researcher has focused to determine a convergent-divergent nozzle shape with this design aspect by using a thermo-fluid analytical approach. This has obviously motivated to carryout the present research work.

In this paper, a modified classical analysis for the nozzle flow based on an analytical approach has been described for an adiabatic flow condition with friction. This analysis has been developed for the consideration of the nozzle efficiency as a design parameter for actual flow events. All the actual effects can be included to the nozzle efficiency term by reducing its magnitude. However, the exact value of the nozzle efficiency can be determined experimentally. Therefore, this paper focuses on the frictionally resisted flow and effects of thermodynamic properties like, pressure, temperature, enthalpy, entropy, etc. The main objective of this work is to determine an actual index of expansion for fluid flow and to investigate the effect of area on Mach number in the flow field. Next, it is thermodynamically established that there always exists a subsonic flow at a downstream from the throat in the divergent part. On the other hand, a supersonic velocity can be achieved at the exit of the convergent-divergent nozzle. Finally, a convergent-divergent nozzle shape has been determined under an actual flow situation by using a reference state.

2. MATHEMATICAL ANALYSIS

The velocity of the fluid increases continuously in the nozzle with an expense of pressure drops in the flow direction. The increase in kinetic energy is a function of enthalpy change in compressible fluids. The following assumptions are made to establish the present analytical model:

(i) The flow is one-dimensional and steady state.

(ii) The flow is adiabatic, i.e. there is no heat transfer taking place between fluid and surrounding.

(iii) The nozzle efficiency is a constant through out the expansion process. For frictional and other losses, nozzle efficiency diminishes.

2.1 Determination of an Actual Index of Expansion

Let there be a very large number of small stages during the expansion of fluid in a convergent-divergent nozzle (Fig. 1). It is assumed that there is an infinitesimally small drop of pressure (Δp) in each small stage. The whole expansion of fluid in a nozzle is shown in h-s diagram as depicted in Fig. 2, where the state points 1 and $4'$ are the inlet and exit conditions of the nozzle, respectively. Due to the irreversibility present in the flow, it causes to increase the entropy of the fluid. As N number of small stages for the expansion is considered, pressure drop in each stage is

$$\Delta p = \frac{p_4 - p_1}{N} = p_b - p_a \tag{1}$$

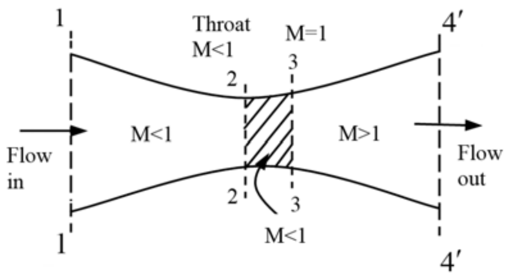

Fig. 1. Convergent-divergent nozzle and Mach number of flow in different sections.

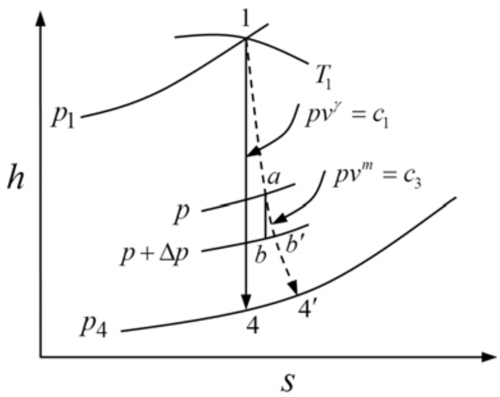

Fig. 2. Adiabatic expansion process in a nozzle represented on *h-s* diagram.

where, p_1 and p_4 are initial and final pressure in any expansion of fluid in a nozzle. The change in specific enthalpy (Fig. 2) in a stage isentropically can be written from the momentum equation as

$$h_b - h_a = v\Delta p \tag{2}$$

Now, an actual change in enthalpy is a function of nozzle efficiency (η_n) and can be expressed as follows:

$$h_b' - h_a = \eta_n v \Delta p \tag{3}$$

where, h_b' is the actual enthalpy at the outlet of a stage. Combining Eqs. (2) and (3) yields

$$h_b' - h_b = -v\Delta p(1-\eta_n) \tag{4}$$

From the definition of enthalpy, one can write

$$h_b' = u_b' + (p + \Delta p) v_b' \tag{5}$$

and

$$h_b = u_b + (p + \Delta p) v_b \tag{6}$$

From Eqs. (5) and (6),

$$h_b' - h_b = u_b' - u_b + (p + \Delta p)\left(v_b' - v_b\right) \tag{7}$$

Equation (7) can be expressed in differential form as

$$h_b' - h_b = \Delta u + p\ \Delta v \tag{8}$$

As $\Delta p \Delta v$ is very small, it is neglected. From Eqs. (4) and (8), one can obtain,

$$\Delta u + p\ \Delta v = -v\Delta p(1-\eta_n) \tag{9}$$

Again from the first law of thermodynamics,

$$\delta q = du + pdv \tag{10}$$

As the adiabatic flow is taking place, there is no heat exchange across the nozzle boundary, i.e. $\delta q = 0$. Therefore, Eq. (10) becomes

$$du = -pdv \tag{11}$$

Here it is pointed out that the change in thermodynamic properties between two state points does not depend upon the path to be followed. Hence Eq. (11) can be integrated by adopting the isentropic process $pv^\gamma = c_1$ to obtain the specific internal energy expression as

$$u = -\frac{pv}{1-\gamma} + c_2 \tag{12}$$

where, c_1 and c_2 are constants. To eliminate the constant c_2 from Eq. (12), the differentiation has been made:

$$\Delta u = -\frac{(p\Delta v + v\Delta p)}{1-\gamma} \tag{13}$$

Equation (13) is substituted into Eq. (9) to obtain a pressure-volume relationship,

$$\frac{(p\Delta v + v\Delta p)}{1-\gamma} - p\ \Delta v = v\Delta p(1-\eta_n) \tag{14}$$

Now Eq. (14) can be rearranged as

$$\frac{\Delta p}{p}+\frac{\gamma}{1-(1-\eta_n)(1-\gamma)}\frac{\Delta v}{v}=0 \quad (15)$$

Using calculus, Eq. (15) is expressed as

$$\frac{dp}{p}+\frac{\gamma}{1-(1-\eta_n)(1-\gamma)}\frac{dv}{v}=0 \quad (16)$$

Integrating Eq. (16) yields the following relationship:

$$pv^m=c_3 \quad (17)$$

where

$$m=\frac{\gamma}{1-(1-\eta_n)(1-\gamma)} \quad (18)$$

and c_3 is an integration constant. Here it may be noted that although the actual expansion in the nozzle is an irreversible, this process is represented by a thermodynamic relationship i.e., $pv^m = c_3$ which has been derived under simplifying assumptions.

2.2 Area and Mach Number Relationship

The fluid motion is governed by the fundamental equations. Considering the constant mass flow rate $(\dot{w})$ under steady state, the conservation of mass equation for 1-D flow becomes

$$\frac{AC}{v}=\dot{w} \quad (19)$$

where, A and C are the local cross-sectional area and fluid velocity at that section, respectively. Rewriting Eq. (19) in a differential form as

$$\frac{dA}{A}=\frac{dv}{v}-\frac{dC}{C} \quad (20)$$

Now, considering the nozzle efficiency, the momentum equation in the flow direction is

$$CdC=-\eta_n v dp \quad (21)$$

From Eq. (21), one can write

$$\frac{dC}{C}=-\eta_n\frac{v}{C^2}dp \quad (22)$$

Now, considering the fluid friction, the actual process equation $\left(pv^m = c_3\right)$ can be written in differential form as

$$\frac{dv}{v}=-\frac{1}{m}\cdot\frac{dp}{p} \quad (23)$$

Putting the mathematical expressions of dv/v and dC/C into Eq. (20) yields

$$\frac{dA}{A}=-\frac{1}{m}\cdot\frac{dp}{p}\left(1-\frac{m\eta_n}{\gamma}\frac{1}{M^2}\right) \quad (24)$$

Equation (24) provides a relationship between area and Mach number in a nozzle flow under an actual design condition for frictionally resisted expansion. The flow conditions in a convergent-divergent nozzle can be determined from the above expression. In the convergent section, dA/A is always negative, hence the flow velocity is always subsonic $\left(M<\sqrt{m\eta_n/\gamma}\right)$. At the throat section, $dA/A=0$, fluid velocity at this section is also subsonic, i.e. $M=\sqrt{m\eta_n/\gamma}$. A subsonic velocity which is greater than this Mach number $\left(\sqrt{m\eta_n/\gamma}\right)$ and less than 1 in the divergent part of the nozzle shown in Fig. 1 by hatching lines where dA/A is positive. This portion of the divergent part acts as a nozzle in spite of that the subsonic fluid is there. In this zone, the pressure gradient is negative as well as the velocity gradient is positive. It is important to note that the flow velocity in the small region of the divergent part which is adjacent to the throat section is always subsonic. Unlike the conventional concept, there is no formation of shock waves to maintain a subsonic flow in that part of the divergent section although the nozzle flow characteristics always exist.

2.3 Reference Conditions

In general, stagnation properties of fluid can vary throughout the flow field. However, if the flow is reversible adiabatic, then $h+(1/2)C^2$ is constant at every location in the flow. It follows that the stagnation properties h_0, p_0, T_0, etc., are constant throughout the isentropic flow. In case of nozzle flow, fluid velocity at the inlet is very small so that it is assumed at rest. Then the properties at the inlet of the nozzle are equal to the stagnation properties. The stagnation properties at a point are defined as those which are to be obtained if the local flow where imagined to cease to zero velocity isentropically. However, in an actual case, the process for obtaining zero velocity may not be possible to achieve by the isentropic way. With the consideration of an actual process, properties of fluid can be determined to zero velocity from a local velocity. These may be referred to an actual stagnation value. The momentum equation for nozzle flow with an actual flow condition is

$$d\left(C^2/2\right)=-\eta_n v dp \quad (25)$$

Integrating Eq. (25) by using Eq. (17) as

$$h_0=\frac{C^2}{2}+\frac{\eta_n m}{m-1}pv \quad (26)$$

From Eq. (26), the stagnation temperature, pressure, and density can be expressed, respectively as

$$\left(\frac{T_0}{T}\right)_{actual}=\eta_n\left(\frac{m}{m-1}\right)\left(\frac{\gamma-1}{\gamma}\right)+\left(\frac{\gamma-1}{2}\right)M^2 \quad (27)$$

$$\left(\frac{p_0}{p}\right)_{actual}$$

$$= \left[\eta_n \left(\frac{m}{m-1} \right) \left(\frac{\gamma-1}{\gamma} \right) + \left(\frac{\gamma-1}{2} \right) M^2 \right]^{\frac{m}{m-1}} \quad (28)$$

and

$$\left(\frac{\rho_0}{\rho} \right)_{actual} = \left[\eta_n \left(\frac{m}{m-1} \right) \left(\frac{\gamma-1}{\gamma} \right) + \left(\frac{\gamma-1}{2} \right) M^2 \right]^{\frac{1}{m-1}} \quad (29)$$

Equations (27)–(29) are different from the stagnation properties with consideration of the reversible adiabatic flow condition (Fox *et al.*, 2004). Therefore, these may be important design parameters determined in the present study under an actual flow condition.

In addition to the stagnation properties, there is another useful set of reference quantities. These are called sonic or critical conditions and are denoted by an asterisk. From the continuity equation, area relation for a nozzle flow with considering the critical condition can be written as

$$\frac{A}{A^*} = \frac{1}{M} \frac{\rho^*}{\rho} \sqrt{\frac{T^*}{T}} \quad (30)$$

Using Eqs. (25) - (29), Eq. (30) can be expressed as

$$\frac{A}{A^*} = \frac{1}{M} \left[\frac{\eta_n \left(\frac{m}{\gamma} \right) \left(\frac{\gamma-1}{m-1} \right) + \frac{\gamma-1}{2} M^2}{\eta_n \left(\frac{m}{\gamma} \right) \left(\frac{\gamma-1}{m-1} \right) + \frac{\gamma-1}{2}} \right]^{\frac{m+1}{2(m-1)}} \quad (31)$$

For an isentropic flow the area of the passage A^* at which the sonic conditions are attained (Fox *et al.*, 2004), is given by

$$\frac{A}{A^*} = \frac{1}{M} \left[\frac{1 + \frac{\gamma-1}{2} M^2}{1 + \frac{\gamma-1}{2}} \right]^{\frac{\gamma+1}{2(\gamma-1)}} \quad (32)$$

From a comparison of Eqs. (31) and (32), a reference condition of an actual flow is dependent on, in addition to, nozzle efficiency and actual index of expansion. Therefore, it is to highlight in the present study that the reference condition for an actual process has been established first time for the compressible fluid flow in the present paper. From the above mathematical relationship between area and Mach number, the velocity in convergent and divergent nozzles has been identified under an actual flow condition. Here it can be pointed that an actual shape of nozzle with the variation of Mach number is determined first time.

3. RESULTS AND DISCUSSION

Figure 3 shows the variation of dA/A with Mach number in a convergent-divergent nozzle for actual and ideal flow conditions. It is obvious from the figure that, in the flow conditions with 100% nozzle efficiency, the flow follows an ideal pattern as explained in many text books (Kearton, 2004; Fox *et al.*, 2004). It is observed that, for a specific fluid, though the isentropic index of expansion is a constant, the nozzle efficiency is influenced by the flow pattern. The fluid flows with subsonic speed after leaving the throat area as shown in this figure for $\eta_n = 0.6$ and $\eta_n = 0.8$. Therefore, there is a subsonic velocity of the fluid in the first part of the divergent section adjacent to the throat for the nozzle efficiency less than 100%. For the actual analysis, energy losses are encountered in the nozzle and as a result, nozzle efficiency decreases always from the maximum value (100%). With the consideration of this situation, it can be observed that the velocity at the inlet to the divergent section of a convergent-divergent nozzle is always subsonic in order to accelerate the flow in the divergent section to obtain a supersonic velocity at the exit of the nozzle. It is thus obvious that the length and the cross-sectional area of the divergent section are responsible to have subsonic, sonic or supersonic velocity at the outlet of a convergent-divergent nozzle. The correctness of the present analysis can be verified by selecting $\eta_n = 1$. This consideration becomes nozzle flow isentropic. For this design situation, the area-Mach number curve has been depicted in Fig. 3 and the sonic state has been obtained at the throat. It implies that the analysis presented in this study is accurate and correct.

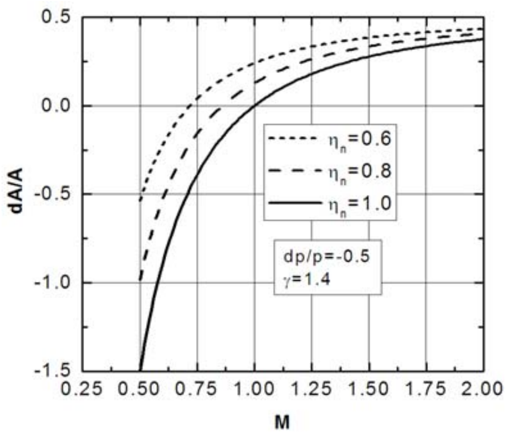

Fig. 3. dA/A with Mach number in a convergent-divergent nozzle for actual and ideal flow conditions.

Figure 4 depicts the effect of actual index of expansion on a relationship between dA/A and Mach number. It is observed that, though the efficiency of the nozzle is constant, the change in pressure and temperature in the expansion process has an effect on fluid flow. The effect of the index of expansion on dA/A and Mach number is insignificant. Further, the Mach number is around 0.8 at the throat for the supersonic flow in the divergent section. The unity of Mach number (Bayt, 1999) is not at the throat location, but it is found a little bit downstream from the throat, i.e. in the small region of the divergent section. This phenomenon was shown by Tsuge (2015) with a rigorous mathematical approach. The main difference of the present work from the published work (Tsug, 2015) is the method of analysis. The present study has been carried out based on the analytical approach with satisfying all the basic laws for thermodynamics and fluid flows.

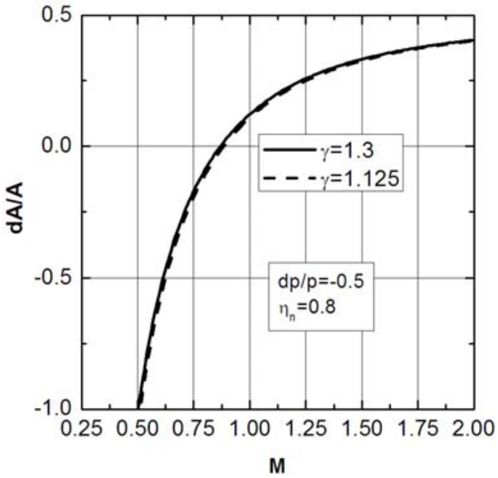

Fig. 4. Effect of actual index of expansion on a relationship between dA/A and Mach number.

Figure 5 shows a dimensionless shape of convergent-divergent nozzle with the variation of Mach number in an actual flow for the superheated steam at the inlet condition. In comparison, this variation for an isentropic flow has been plotted. Each curve has double valued of Mach number at a constant A/A^* which are in the convergent part as well as in the divergent section. From the reference condition analysis, in case of actual flow condition, a subsonic velocity is at the throat and the subsonic velocity still exists in the divergent part adjacent to the throat although it is an accelerated flow and the sonic velocity is at a divergent part of the nozzle. On the other hand, for an isentropic flow, sonic velocity is always at the minimum cross section of the nozzle. This observation has already been demonstrated from the previous results of the present study. From these design curves plotted in Fig. 5, it can also be highlighted that the dimensionless area requirement for the flow maintaining the same velocity in the convergent section is always lesser for an actual flow than that for an ideal flow. However, this trend requirement is opposite in the divergent section. Therefore, the present study will help to determine the actual nozzle shape under a practical design circumstance. It can also be demonstrated that no such investigation is available to find this observation for flow in nozzles subject to friction and adiabatic conditions.

4. CONCLUSIONS

In this paper, the effect of fluid friction on the flow velocity is studied by using the basic laws of conservation of mass, momentum and energy. It is observed that due to the fluid friction entropy increases but the velocity of the fluid at the throat decreases and it results an existence of subsonic flow to a downstream from the throat. In a region of divergent section in the convergent-divergent nozzle, a subsonic velocity always exists under a frictional flow condition. So from this study, it is highlighted that there is no mandatory requirement of the sonic velocity at the throat for obtaining a supersonic velocity at the exit of a convergent-divergent nozzle. A transonic velocity is found in the first part of the divergent section of the nozzle in an actual flow condition. Depending upon the length of the divergent and cross sectional area, fluid velocity at the exit may be subsonic, sonic or supersonic. The present study indicates that the convergent-divergent nozzle can be used for any value of fluid velocity required at the exit and this observation has been analytically established. The reference flow conditions newly derived are also dependent on the nozzle efficiency and actual index of expansion. The actual nozzle shape has been determined. Therefore, the present study may be helpful to a designer to know the actual design information.

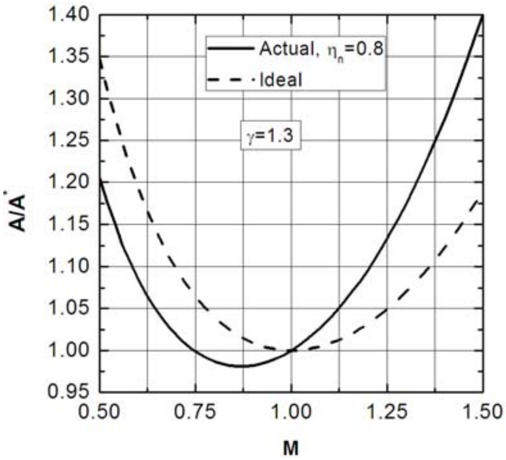

Fig. 5. Variation of A/A^* with M in actual and isentropic flow for $\gamma = 1.3$.

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