occur in many industrial and natural flows such as those in the food, cosmetics, pharmaceutical and petroleum industries, bioengineering, geothermal energy utilization, carbon dioxide geologic sequestration, construction of oil wells and mud flows. In polymer processing the melt flows at high temperature before the extrusion process and in such situations understanding of convection processes in viscoelastic fluids and in particular control of instabilities becomes important in order to avoid the appearance of non-homogeneities in the final product. Hence, it is very essential to know the convective behavior in viscoelastic fluids.

The Rayleigh-Bénard instability for viscoelastic fluids has been studied extensively and copious literature is available (Green, 1968; Vest and Arpaci, 1969; Sikolov and Tanner, 1972; Eltayeb, 1977; Rosenblat, 1986; Martinez-Mardones and Perez-Garcia, 1990, 1992; Larson, 1992; Khayat, 1995; Martinez-Mardones *et al*., 1996; Kolodner, 1998; Li and Khayat, 2005; Swamy and Sidram, 2012). An unusual result observed, contrary to the stationary onset in Newtonian fluids, is that the onset of convection in viscoelastic fluids occur via oscillatory convection depending on the fluid elasticity. Using DNA suspensions, Kolodner, (1998) experimentally confirmed this behavior of instability and it was observed in dilute polymer solutions, consisting of a Newtonian solvent and a polymeric solute which is well represented by the Oldroyd-B constitutive model (Bird *et al*. 1987). Hirata *et al*. (2015) examined convective and absolute nature of instabilities in Rayleigh–Bénard–Poiseuille mixed convection for viscoelastic fluids.

Double-diffusive convection in a Newtonian fluid layer has been studied extensively, both experimentally and theoretically, because of its wide range of applications in many fields of science and engineering (for details see Chen and Johnson, 1984). Excellent reviews on the development of this subject are reported (Turner, 1973, 1974, 1985; Huppert and Turner, 1981; Platten and Legros, 2011). But the consideration of binary fluid aspects in the study of convective instability in viscoelastic fluids seems to be pertinent as these fluids involve solvent and solute. Kolodner (1998) also attributed that the discrepancy between his experimental prediction and theoretical results could be due to the neglect of binary fluid aspects with viscoelastic properties in the theoretical analysis. Nonetheless, the study has received only limited attention in the literature. Martinez-Mardones *et al*. (2000) performed a weakly nonlinear stability analysis for stationary convection in a binary Oldroyd fluid considering Soret effect. Malashetty and Swamy (2010) discussed linear and a weakly nonlinear double-diffusive convection in a viscoelastic fluid layer. Ashraf *et al*. (2016) investigated mixed convection flow of an Oldroyd-B fluid over a stretching surface with convective boundary conditions. Double diffusive convection in a viscoelastic fluid-saturated porous layer is also discussed (Malashetty *et al*. 2009; Awad *et al*. 2010; Wang and Tan, 2008, 2011; Kumar and Bhadauria, 2011; Malashetty and Biradar, 2011; Altawallbeh *et al*. 2017).

The intent of the present paper is to investigate nonlinear stationary and oscillatory stability of double-diffusive Oldroyd-B fluid layer. The perturbation method is used to perform the nonlinear stability analysis and the stability of bifurcating stationary and oscillatory solutions is analyzed by deriving cubic Landau equations. Such a study is of interest to know if subcritical instabilities will occur and under what conditions in terms of the parameters of the problem. The results of linear stability theory are also given as the nonlinear stability analysis is based on these results. Besides, convective rates of heat and mass transfer are estimated in terms of Nusselt numbers.

2. PROBLEM FORMULATION

We consider an incompressible double diffusive Oldroyd-B fluid layer of thickness d in the finite vertical z - direction and extending to infinity in the x and y directions in the presence of gravity. The lower boundary at $z=0$ and upper boundary at $z=d$ are maintained at constant but different temperatures T_0 and $T_0-\Delta T$ $(\Delta T>0)$, and solute concentrations S_0 and $S_0-\Delta S$ $(\Delta S>0)$, respectively. The density ρ is assumed to vary linearly with temperature T and solute concentration S in the form

$$\rho=\rho_0\left(1-\alpha_T\left(T-T_0\right)+\alpha_S\left(S-S_0\right)\right) \tag{1}$$

where α_T is the coefficient of thermal expansion, α_S is the coefficient of solute expansion and ρ_0 is the reference density. The nonlinear constitutive equation for an Oldroyd-B fluid is (Rosenblat, 1986; Bird *et al*. 1987)

$$\underset{\sim}{\tau}+\lambda_1\left(\frac{\partial \underset{\sim}{\tau}}{\partial t}+(\vec{q}\cdot\nabla)\underset{\sim}{\tau}-(\nabla\vec{q})^T\underset{\sim}{\tau}-\underset{\sim}{\tau}(\nabla\vec{q})\right)= \mu\left(\underset{\sim}{A}+\lambda_2\left(\frac{\partial \underset{\sim}{A}}{\partial t}+(\vec{q}\cdot\nabla)\underset{\sim}{A}-(\nabla\vec{q})^T\underset{\sim}{A}-\underset{\sim}{A}(\nabla\vec{q})\right)\right) \tag{2}$$

where $\underset{\sim}{\tau}$ is the stress tensor, $\vec{q}=(u,v,w)$ is the velocity, $\underset{\sim}{A}=\nabla\vec{q}+(\nabla\vec{q})^T$ is the rate-of-strain tensor, μ is the fluid viscosity, λ_1 is the relaxation time, λ_2 is the retardation time and it is noted that $\lambda_2\le\lambda_1$. It is well-known that the constitutive equation considered includes Stokes's law adopted in the theory of Newtonian viscous fluid flows as a particular case for $\lambda_1=\lambda_2$ and to the Maxwell fluid when $\lambda_2=0$. Under the Boussinesq approximation, the governing equations for the problem under consideration are

$$\nabla\cdot\vec{q}=0 \tag{3}$$

$$\rho_0\left(\frac{\partial\vec{q}}{\partial t}+(\vec{q}\cdot\nabla)\vec{q}\right)=-\nabla p+\nabla\cdot\underset{\sim}{\tau}+\rho\vec{g} \tag{4}$$

$$\frac{\partial T}{\partial t}+(\vec{q}\cdot\nabla)T=\kappa_T\nabla^2 T \tag{5}$$

$$\frac{\partial S}{\partial t}+(\vec{q}\cdot\nabla)S=\kappa_S\nabla^2 S \tag{6}$$

where p is the pressure, $\vec{g}=(0,0,g)$ is the acceleration due to gravity, κ_T is the effective thermal diffusivity and κ_S is the solute analog of κ_T. The basic state is found to be

$$\vec{q}_b=0, T_b=T_0-\frac{\Delta T}{d}z, S_b=S_0-\frac{\Delta S}{d}z,$$
$$p_b=p_0-\rho_0\, g\left(z+\alpha_T\frac{\Delta T}{2d}z^2-\alpha_S\frac{\Delta S}{2d}z^2\right) \tag{7}$$

where p_0 is the pressure at $z=0$. The finite amplitude perturbations on the basic state are superposed in the form:

$$\vec{q}=\vec{q}_b+\vec{q}', p=p_b+p', \rho=\rho_b+\rho'$$
$$\underset{\sim}{\tau}=\underset{\sim}{\tau}_b+\underset{\sim}{\tau}', T=T_b+T', S=S_b+S' \tag{8}$$

where primes indicate perturbations over their equilibrium counterparts. Equation (8) is substituted back into the governing equations and the perturbation equations are written in the dimensionless form by introducing the scaling d, d^2/κ_T, κ_T/d, $\mu\kappa_T/d^2$, ΔT and ΔS as reference quantities for length, time, perturbation velocity, stress components, temperature and solute concentration, respectively. The pressure is then eliminated by operating curl once and the stream function $\psi(x,z,t)$ is introduced in the form

$$u'=\partial\psi/\partial z\,,\; w'=-\partial\psi/\partial x\,. \tag{9}$$

to obtain the stability equations in the form (after neglecting the primes)

$$\frac{1}{Pr}\mathscr{X}\left(\nabla^2\psi\right)+\left(R_T T-R_S\,S\right)_{,x}-N=0 \tag{10}$$

$$\mathscr{X}T+\psi_{,x}-\nabla^2 T=0 \tag{11}$$

$$\mathscr{X}S+\psi_{,x}-\Gamma\,\nabla^2 S=0 \tag{12}$$

$$\underset{\sim}{\tau}+\Lambda_1\left(\frac{\partial\underset{\sim}{\tau}}{\partial t}+(\vec{q}\cdot\nabla)\underset{\sim}{\tau}-(\nabla\vec{q})^T\,\underset{\sim}{\tau}-\underset{\sim}{\tau}\,(\nabla\vec{q})\right)=$$
$$\underset{\sim}{A}+\Lambda_2\left(\frac{\partial\underset{\sim}{A}}{\partial t}+(\vec{q}\cdot\nabla)\underset{\sim}{A}-(\nabla\vec{q})^T\,\underset{\sim}{A}-\underset{\sim}{A}\,(\nabla\vec{q})\right) \tag{13}$$

where $Pr=\nu/\kappa_T$ is the Prandtl number, $R_T=\alpha_T\, g\, d^3\Delta T/\nu\kappa_T$ is the thermal Rayleigh number, $R_S=\alpha_T\, g\, d^3\Delta S/\nu\kappa_T$ is the solute Rayleigh number, $\Gamma=\kappa_{S1}/\kappa_T$ is the diffusivity ratio, $\Lambda_1=\lambda_1\kappa_T/d^2$ is the relaxation parameter, $\Lambda_2=\lambda_2\kappa_T/d^2$ is the retardation parameter, $\mathscr{X}=\partial/\partial t+J(\cdot\,,\psi)$ is the nonlinear differential operator,

$N=(\tau_{xx}-\tau_{zz})_{,xz}+\tau_{xz,zz}-\tau_{xz,xx}$ and $\nabla^2=\partial^2/\partial x^2+\partial^2/\partial z^2$.

The boundaries are rigid stress-free and perfect conductors of heat and solute concentration. The corresponding boundary conditions are:

$$\psi=\psi_{,zz}=T=S=0 \text{ at } \; z=0,1\,. \tag{14}$$

In components, Eq. (13) can be expressed as

$$\tau_{xz}+\Lambda_1\left(\mathscr{X}\left(\tau_{xz}\right)+\frac{1}{2}\nabla^2\psi\; U-\frac{1}{2}\Delta_1\psi\; V\right)=$$
$$\Delta_1\psi+\Lambda_2\left(\mathscr{X}\left(\Delta_1\psi\right)+2\psi_{,xz}\nabla^2\psi\right) \tag{15}$$

$$U+\Lambda_1\left(\mathscr{X}(U)-2\nabla^2\psi\;\tau_{xz}-2\psi_{,xz}\;V\right)=$$
$$4\psi_{,xz}+\Lambda_2\left(4\mathscr{X}\left(\psi_{,xz}\right)-2\nabla^2\psi\;\Delta_1\psi\right) \tag{16}$$

$$V+\Lambda_1\left(\mathscr{X}(V)-2\psi_{,xz}\,U-2\Delta_1\psi\;\tau_{xz}\right)=$$
$$-2\Lambda_2\left(4\left(\psi_{,xz}\right)^2+\left(\Delta_1\psi\right)^2\right) \tag{17}$$

where

$U=\tau_{xx}-\tau_{zz}$, $V=\tau_{xx}+\tau_{zz}$ and

$\Delta_1\psi=\psi_{,zz}-\psi_{,xx}$.

3. NONLINEAR STABILITY ANALYSIS

The nonlinear stability analysis is carried out using the perturbation method (Rosenblat, 1986; Venezian. 1969). Accordingly, a small bifurcation parameter χ that indicates deviation from the critical state is introduced and the dependent variables $\Phi=(\psi,T,S,U,V,\tau_{xz})$ and as well as R_T are expanded in powers of χ in the form

$$\Phi=\Phi_1\,\chi+\Phi_2\,\chi^2+\Phi_3\,\chi^3+\cdots$$
$$R_T=R_{Tc}+\chi^2\,R_{T2}+\cdots \tag{18}$$

where R_{Tc} is the critical Rayleigh number chosen depending on the type of bifurcation. Also, a small time scale is introduced in the form $s=\chi^2 t$ and the operator $\partial/\partial t$ is replaced depending on the nature of bifurcating solutions.

3.1. Bifurcation of Steady Solutions

The stationary convective solution that bifurcates from the basic state at $R_T^S=R_{Tc}^S$, where R_{Tc}^S is the critical Rayleigh number for the stationary onset..

Substituting Eq. (18) and $\partial/\partial t = \chi^2 \partial/\partial s$ into Eqs. (10)-(13) and equating the coefficients of $\chi^i\ (i = 1,2,3,...)$ lead to the following sequence of equations

$$\left(R_{Tc} T_i - R_S S_i\right)_{,x} - \mathrm{N}_i = G_{1i} \tag{19}$$

$$\psi_{i,x} - \nabla^2 T_i = G_{2i} \tag{20}$$

$$\psi_{i,x} - \Gamma \nabla^2 S_{1i} = G_{3i} \tag{21}$$

with

$$\mathrm{N}_i = U_{i,xz} + \Delta_1 \tau_{xz}^{(i)} \ \ (i = 1,2,3,\cdots) \tag{22}$$

where $G_{1i} - G_{3i}$ are the nonlinear terms which are determined using previous order solutions. The equations for the stress components have the forms

$$\begin{aligned} &\tau_{xz}^{(i)} - \Delta_1 \psi_i = X_i \\ &U_i - 4\psi_{i,xz} = Y_i \\ &V_i = Z_i \end{aligned} \tag{23}$$

where $X_i - Z_i$ are the nonlinear terms which are determined using previous order solutions.

The boundary conditions are

$$T_i = S_i = \psi_i = \psi_{i,zz} = \tau_{xz}^{(i)} = U_{i,z} = 0 \text{ at } z = 0,1 . \tag{24}$$

The first order solution corresponding to $R_{Tc} = R_{Tc}^S$ is

$$\begin{aligned} &(T_1, S_1, U_1, V_1) = (A_1, C_1, D_1, 0)\cos\alpha x \cos\pi z \\ &(\psi_1, \tau_{xz}^{(1)}) = (B_1, E_1)\sin\alpha x \sin\pi z \end{aligned} \tag{25}$$

The undetermined amplitudes are related by

$$\begin{aligned} &A_1 = -\frac{\alpha}{\delta^2} B_1, C_1 = -\frac{\alpha}{\delta^2 \Gamma} B_1, D_1 = 4\pi\alpha B_1, \\ &E_1 = -cB_1, R_T^S = \frac{R_S}{\Gamma} + \frac{\delta^6}{\alpha^2} \end{aligned} \tag{26}$$

where $c = \pi^2 - \alpha^2$ and the amplitude B_1 remains undetermined at this stage.

We note that R_T^S attains minimum value at $\alpha_c = \pi/\sqrt{2}$ and the critical value is

$$R_{Tc}^S = \frac{R_S}{\Gamma} + \frac{27\pi^4}{4} \tag{27}$$

which is independent of viscoelastic effects and coincides for the Newtonian fluid (Veronis, 1968). Following the standard nonlinear stability analysis procedure, a cubic Landau equation is derived from the third order equation in the form

$$\gamma B_1' = \alpha^2 R_{T2} B_1 - k B_1^3 \tag{28}$$

where

$$\gamma = \frac{\alpha^2}{\delta^2} R_{Tc} - \frac{\alpha^2}{\delta^2 \Gamma^2} R_S + \left(Pr^{-1} - \delta^2\right)\delta^4 \tag{29}$$

$$k = \frac{\alpha^4}{8\delta^2} R_{Tc} - \frac{\alpha^4}{8\delta^2 \Gamma^3} R_S - M\delta^2 \tag{30}$$

$$M = \frac{1}{16}\Lambda_1\left(\Lambda_1 - \Lambda_2\right)\left(m - n\right) \tag{31}$$

with

$$m = 9\left(\pi^4 + \alpha^4\right)^2 + 4\pi^2\alpha^2\left(\pi^4 + \alpha^4\right) + 36\pi^4\alpha^4$$

$$n = 9c^4 + 8\pi^2\alpha^2c^2 + 144\pi^4\alpha^4 ,$$

$$c^2 = \pi^2 - \alpha^2 .$$

The steady state amplitude exists in the following form:

$$B_1^2 = \frac{8\delta^2}{\alpha^2}\frac{R_{T2}}{\Omega} \tag{32}$$

where

$$\Omega = R_{Tc} - \frac{R_S}{\Gamma^3} - \frac{8\delta^4}{\alpha^4} M . \tag{33}$$

Although the stationary onset is independent of viscoelastic parameters, from Eq. (33) it is seen that these parameters influence the stability of stationary bifurcation. The stationary bifurcation is supercritical (stable) if $\Omega > 0$ and subcritical (unstable) if $\Omega < 0$. From Eq. (33) it is evident that, if $R_S = 0$ (single component system) the stationary solution always bifurcates supercritically. For other choices of physical parameters, subcritical bifurcation is possible.

For this case, the heat and mass transfer are determined in terms of thermal and solute Nusselt numbers, respectively. The thermal Nusselt number Nu_T and the solute Nusselt number Nu_S are defined as

$$Nu_T = 1 + 2\frac{\left(R_T - R_{Tc}\right)}{\left(R_{Tc} - \frac{R_S}{\Gamma^3}\right) - \frac{8\delta^4}{\alpha^4}\mathrm{M}} \tag{34}$$

$$Nu_S = 1 + \frac{2}{\Gamma^2}\frac{\left(R_T - R_{Tc}\right)}{\left(R_{Tc} - \frac{R_S}{\Gamma^3}\right) - \frac{8\delta^4}{\alpha^4}\mathrm{M}} \tag{35}$$

In the absence of convection, the heat and mass transfer is purely by conduction and in that case $Nu_T = 1 = Nu_S$.

3.2. Bifurcation of Periodic Solutions

The bifurcation of the basic state at the value of

$R_T = R_{Tc}^o$, where R_{Tc}^o is the critical Rayleigh number for the oscillatory onset, can be determined by modifying slightly the method applied in the previous section. The time derivative is not zero in the present case and $\partial/\partial t$ is replaced by $\partial/\partial t + \chi^2 \partial/\partial s$. Noting this fact and substituting Eq. (18) into Eqs. (10)-(12) and equating the coefficients of $\chi^i \; (i = 1, 2, 3, ...)$ lead to a sequence of equations

$$Pr^{-1}\nabla^2\psi_{i,t} + \left(R_{Tc}^o T_i - R_S S_i\right)_{,x} - \mathrm{N}_i = H_{1i} \quad (36)$$

$$T_{i,t} + \psi_{i,x} - \nabla^2 T_i = H_{2i} \quad (37)$$

$$S_{i,t} + \psi_{i,x} - \Gamma \nabla^2 S_i = H_{3i} \quad (38)$$

where N_i is given by Eq. (22) and $H_{1i} - H_{3i}$ are nonlinear terms to be determined consecutively. The boundary conditions are given by Eq. (24). The equations for the stress components have the forms

$$\left(1 + \Lambda_1 \frac{\partial}{\partial t}\right)\tau_{xz}^{(i)} - \left(1 + \Lambda_2 \frac{\partial}{\partial t}\right)\Delta_1 \psi_i = X_i \quad (39)$$

$$\left(1 + \Lambda_1 \frac{\partial}{\partial t}\right)U_i - 4\left(1 + \Lambda_2 \frac{\partial}{\partial t}\right)\psi_{i,xz} = Y_i \quad (40)$$

$$\left(1 + \Lambda_1 \frac{\partial}{\partial t}\right)V_i = Z_i \, . \quad (41)$$

The quantities $X_i - Z_i$ are to be determined consecutively. At first order, the equations are linear and homogeneous whose solution corresponding to $R_{Tc} = R_{Tc}^o$ is

$$\begin{aligned}
T_1 &= \left(A_1 e^{i\omega t} + \bar{A}_1 e^{-i\omega t}\right)\cos\alpha x \cos\pi z \\
\psi_1 &= \left(B_1 e^{i\omega t} + \bar{B}_1 e^{-i\omega t}\right)\sin\alpha x \sin\pi z \\
S_1 &= \left(C_1 e^{i\omega t} + \bar{C}_1 e^{-i\omega t}\right)\cos\alpha x \cos\pi z \\
\tau_{xz}^{(1)} &= \left(E_1 e^{i\omega t} + \bar{E}_1 e^{-i\omega t}\right)\sin\alpha x \sin\pi z \\
U_1 &= \left(D_1 e^{i\omega t} + \bar{D}_1 e^{-i\omega t}\right)\cos\alpha x \cos\pi z \\
V_1 &= 0
\end{aligned} \quad (42)$$

where the overbar above a quantity denotes the complex conjugate. The amplitudes $A_1 - E_1$ and $\bar{A}_1 - \bar{E}_1$ are functions of slow time scale s , while ω, α are the critical values associated with $R_{Tc} = R_{Tc}^o$. The undetermined amplitudes are related by

$$B_1 = -\frac{\left(\delta^2 + i\omega\right)}{\alpha} A_1 \, , \, C_1 = \frac{\left(\delta^2 + i\omega\right)}{\left(\delta^2 \Gamma + i\omega\right)} A_1 \, ,$$

$$D_1 = -\frac{4\pi\left(\delta^2 + i\omega\right)\left(1 + i\omega\Lambda_2\right)}{\left(1 + i\omega\Lambda_1\right)} A_1 \, , \quad (43)$$

$$E_1 = \frac{c\left(\delta^2 + i\omega\right)\left(1 + i\omega\Lambda_2\right)}{\alpha\left(1 + i\omega\Lambda_1\right)} A_1 \, ,$$

$$-i\omega\delta^2 Pr^{-1} - \frac{\left(1 + i\omega\Lambda_2\right)}{\left(1 + i\omega\Lambda_1\right)}\delta^4 + \frac{\alpha^2 R_T^o}{\left(\delta^2 + i\omega\right)} - \frac{\alpha^2 R_S}{\left(\delta^2 \Gamma + i\omega\right)} = 0 \, .$$

From Eq. (43) we find that oscillatory convection sets in at $R_T = R_T^o$, where

$$R_T^o = R_T^S - \omega^2 \left(\frac{\delta^2}{\alpha^2}\left(\frac{1}{Pr} + \delta^2 \frac{\left(\Lambda_1 - \Lambda_2\right)\left(\delta^2 \Lambda_1 - 1\right)}{\left(1 + \Lambda_1^2 \omega^2\right)} \right) + \frac{\left(1 - \Gamma\right)}{\left(\Gamma^2 \delta^4 + \omega^2\right)} \frac{R_S}{\Gamma} \right) \quad (44)$$

with ω^2 satisfying the equation

$$c_1\left(\omega^2\right)^2 + c_2\left(\omega^2\right) + c_3 = 0 \quad (45)$$

where

$$c_1 = \delta^2 Pr \Lambda_1 \Lambda_2 + \Lambda_1^2 \delta^2 \, ,$$

$$c_2 = \left(\Gamma - 1\right) Pr \alpha^2 \Lambda_1^2 R_S + \left(1 + \Gamma^2 \Lambda_1^2 \delta^4\right)\delta^2 + \left(1 + \Gamma^2 \delta^4 \Lambda_1 \Lambda_2\right)\delta^2 Pr + \left(\Lambda_2 - \Lambda_1\right)\delta^2 Pr$$

$$c_3 = \left(\Gamma - 1\right) Pr \alpha^2 R_S + \Gamma^2 \delta^6 + \Gamma^2 \delta^6 Pr + \left(\Lambda_2 - \Lambda_1\right)\Gamma^2 \delta^8 Pr$$

Since $\omega^2 > 0$ for the occurrence of oscillatory convection, a careful glance at Eq. (45) provides the necessary conditions as

$$\Gamma < 1, \; \Lambda_1 - \Lambda_2 > \frac{(1 + Pr^{-1})}{\delta^2} \, . \quad (46)$$

For any chosen parametric values, the critical value of R_T^o with respect to the wave number, denoted by R_{Tc}^o is calculated using the procedure explained in Raghunatha *et al*. (2018).

As in the stationary case, a cubic Landau equation is derived from the third order equations in the form

$$\gamma^* A_1' = \alpha^2 R_{T2} A_1 - \eta^* A_1^2 \bar{A}_1 \quad (47)$$

where

$$\gamma^* = \left(\delta^2 + i\omega\right)\begin{pmatrix} \delta^2 Pr^{-1} - \dfrac{\left(\Lambda_1 - \Lambda_2\right)}{\left(1 + i\omega\Lambda_1\right)^2}\delta^4 + \\ \dfrac{\alpha^2 R_{Tc}^o}{\left(\delta^2 + i\omega\right)^2} - \dfrac{\alpha^2}{\left(\delta^2\Gamma + i\omega\right)^2} R_S \end{pmatrix} \tag{48}$$

$$\eta^* = -\frac{\left(\delta^4+\omega^2\right)}{\alpha^2}\frac{\left(\delta^2+i\omega\right)}{\left(1+i\omega\Lambda_1\right)}M^* + \frac{\left(\delta^4+\omega^2\right)\delta^2\alpha^2 R_{Tc}^o}{4\left(\delta^4+\omega^2\right)} + \frac{\left(\delta^4+\omega^2\right)\pi^2\alpha^2 R_{Tc}^o}{\left(\delta^2+i\omega\right)\left(8\pi^2+4i\omega\right)} + \frac{\left(\delta^4+\omega^2\right)\left(\delta^2+i\omega\right)\delta^2\alpha^2}{4\left(\delta^4\Gamma^2+\omega^2\right)\left(\delta^2\Gamma+i\omega\right)}R_S + \frac{\left(\delta^4+\omega^2\right)\left(\delta^2+i\omega\right)\alpha^2\pi^2}{\left(\delta^2\Gamma+i\omega\right)\left(\delta^2\Gamma+i\omega\right)\left(8\pi^2\Gamma+4i\omega\right)}R_S \tag{49}$$

From Eq. (47) one can obtain

$$\frac{d\left|A_1\right|^2}{ds} = 2R_{T2}\,p_r\left|A_1\right|^2 - 2l_r\left|A_1\right|^4 \tag{50}$$

$$\frac{d\left(ph\left(A_1\right)\right)}{ds} = R_{T2}\,p_i - l_i\left|A_1\right|^2 \tag{51}$$

where $\alpha^2\left(\gamma^*\right)^{-1} = p_r + i\,p_i$, $\eta^*\left(\gamma^*\right)^{-1} = l_r + i\,l_i$ and $ph(\cdot)$ represents the phase shift. The magnitude and direction of the periodic convective solution and also the frequency shift are determined by Eq. (47). We are concerned here about the direction of the bifurcation which depends on the sign of the quantity

$$Q = \frac{p_r}{l_r}R_{T2} = \frac{R_{T2}}{\varphi} \tag{52}$$

(52)

where $\varphi = l_r / p_r$. If $\varphi > 0$ the bifurcation is supercritical and stable and it is subcritical and unstable if $\varphi < 0$. Thus φ is the analog for periodic bifurcations of the quantity Ω, defined by Eq. (33), for steady bifurcations.

For this case, the time and area-averaged thermal Nusselt number $\left(\overline{Nu_T}\right)$ and Solute Nusselt number $\left(\overline{Nu_S}\right)$ are determined and they are given by

$$\overline{Nu_T} = 1 + \frac{\delta^2}{2}\frac{\left(R_T - R_{Tc}^o\right)}{\varphi} \tag{53}$$

$$\overline{Nu_S} = 1 + \frac{\delta^2\left(\delta^4+\omega^2\right)}{2\left(\delta^4\Gamma^2+\omega^2\right)}\frac{\left(R_T - R_{Tc}^o\right)}{\varphi}. \tag{54}$$

4. RESULTS AND DISCUSSION

Double-diffusive nonlinear stationary and oscillatory convection in an Oldroyd-B fluid layer has been investigated using a perturbation method. The stability of bifurcating stationary and oscillatory solutions is discussed by deriving the cubic Landau equations. The criterion for the onset of stationary and oscillatory convection is given as the nonlinear stability analysis for these cases is based on the results of linear instability theory. Both stationary and oscillatory onset critical thermal Rayleigh numbers coincide at well-defined parametric conditions and consequently, a codimension-two bifurcation occurs. The results so obtained are illustrated on a $(\Lambda_2/\Lambda_1, \Lambda_1)$ - plane in Fig. 1 for different values of Pr when $\Gamma = 0.3$ and $R_S = 90$. The region above each curve corresponds to the system unstable under oscillatory convection and below which the system is unstable under stationary convection. For a fixed value of Λ_2, the value of Λ_1 at which codimension-two bifurcation occurs decreases with increasing Pr.

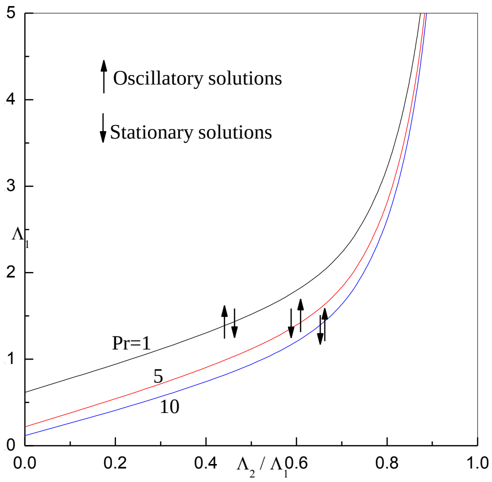

Fig. 1. Loci of codimension-two bifurcation points in the viscoelastic parameters plane for $\Gamma = 0.3$, $R_S = 90$.

Figures 2(a) and 2(b) show the variation of Ω as a function of R_S for different values of Λ_1 with $\Lambda_2 = 0.1$ and Λ_2 with $\Lambda_1 = 0.6$, respectively which correspond to $\Gamma = 0.1$ .These figures demonstrate the possibility of occurring subcritical stationary bifurcation for a range of parametric values indicating the occurrence of instability before the linear threshold is reached. This is probable, because the linear instability analysis provides only sufficient condition for instability. In Fig. 2(b), the curve for $\Lambda_1 = \Lambda_2$ corresponds to the case of Newtonian fluids and $\Lambda_2 = 0$ for the Maxwell fluid. The figure also shows that for the Newtonian and Maxwell fluids the subcritical bifurcation occurs at lower and higher values of solute Rayleigh number R_s , respectively. Besides, the viscoelastic parameters

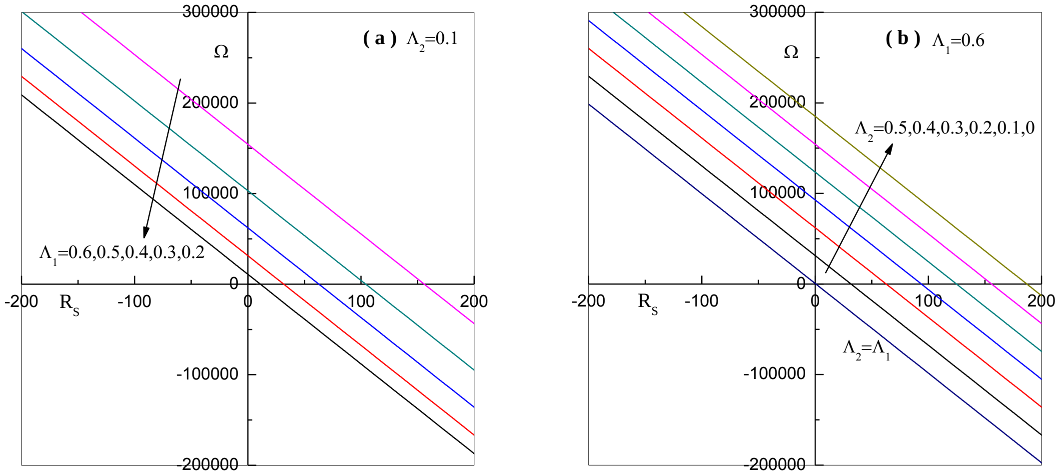

Fig. 2. Regions of supercritical and subcritical steady bifurcations for different values of(a) Λ_1 **(b)** Λ_2 **for** $\Gamma = 0.1$**.**

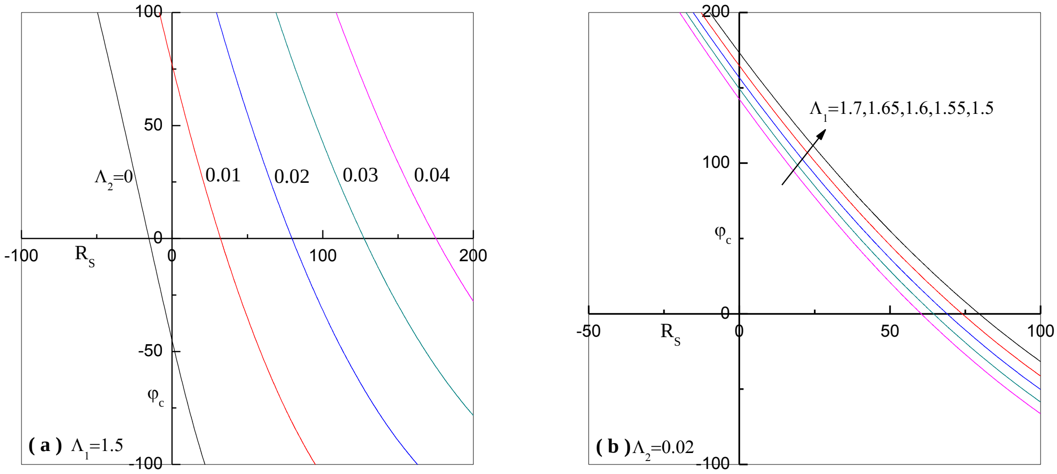

Fig. 3. Regions of supercritical and subcritical oscillatory bifurcations for different values of (a) Λ_2 **(b)** Λ_1 **for** $\mathrm{Pr} = 10, \Gamma = 0.9$**.**

exhibit opposing contributions on the range of R_s beyond which the subcritical bifurcation is possible.

The oscillatory supercritical/subcritical bifurcation can be determined from the sign of φ and for this the critical values of φ are determined for various values of physical parameters. The critical values φ_c are accomplished by substituting α_c and ω_c obtained from the linear instability theory for prescribed parameters in the expression of φ. The trend of φ_c versus R_s for different values of Λ_2, Λ_1 and Pr are presented in Figs. 3(a), 3(b) and 4, respectively. From the figures it is observed that the bifurcation of non-trivial equilibrium oscillatory solution becomes subcritical at higher values of R_s with increasing Λ_2 and Pr, while an opposite trend could be seen with increasing Λ_1.

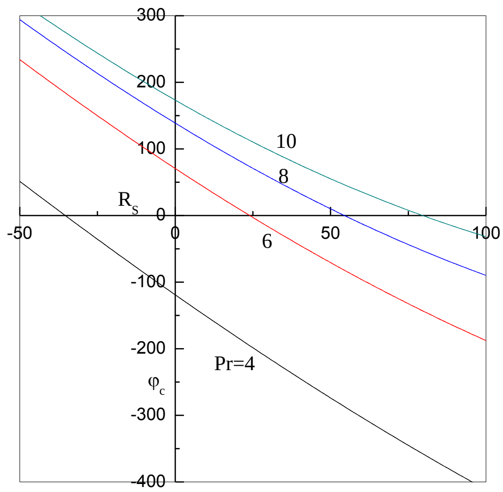

Fig. 4. Regions of supercritical and subcritical oscillatory bifurcations for different values of Pr **for** $\Lambda_1 = 1.5$ **,** $\Lambda_2 = 0.02$ **,** $\Gamma = 0.9$ **.**

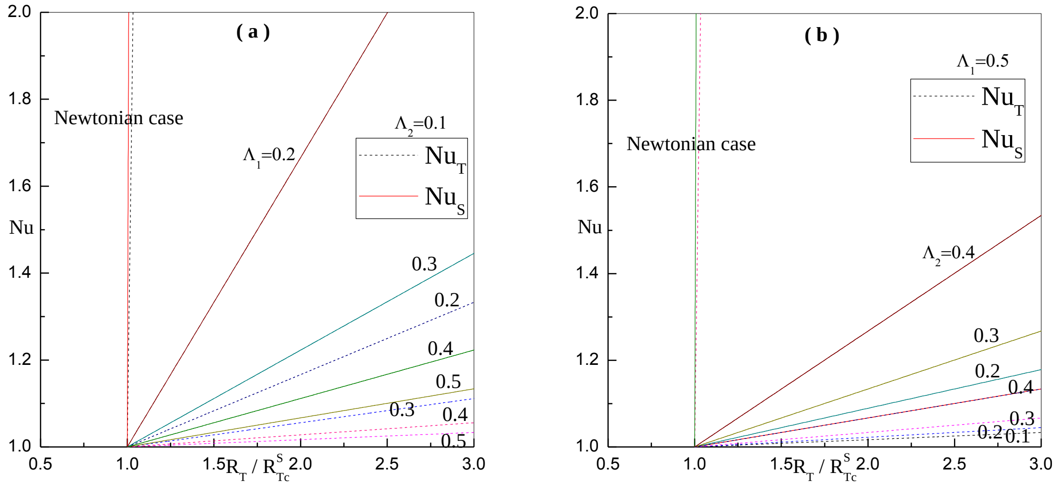

Fig. 5. Area-averaged Nusselt numbers Nu_T and Nu_S for different values of (a) Λ_1 (b) Λ_2 when $\Gamma = 0.5$, $R_S = 100$.

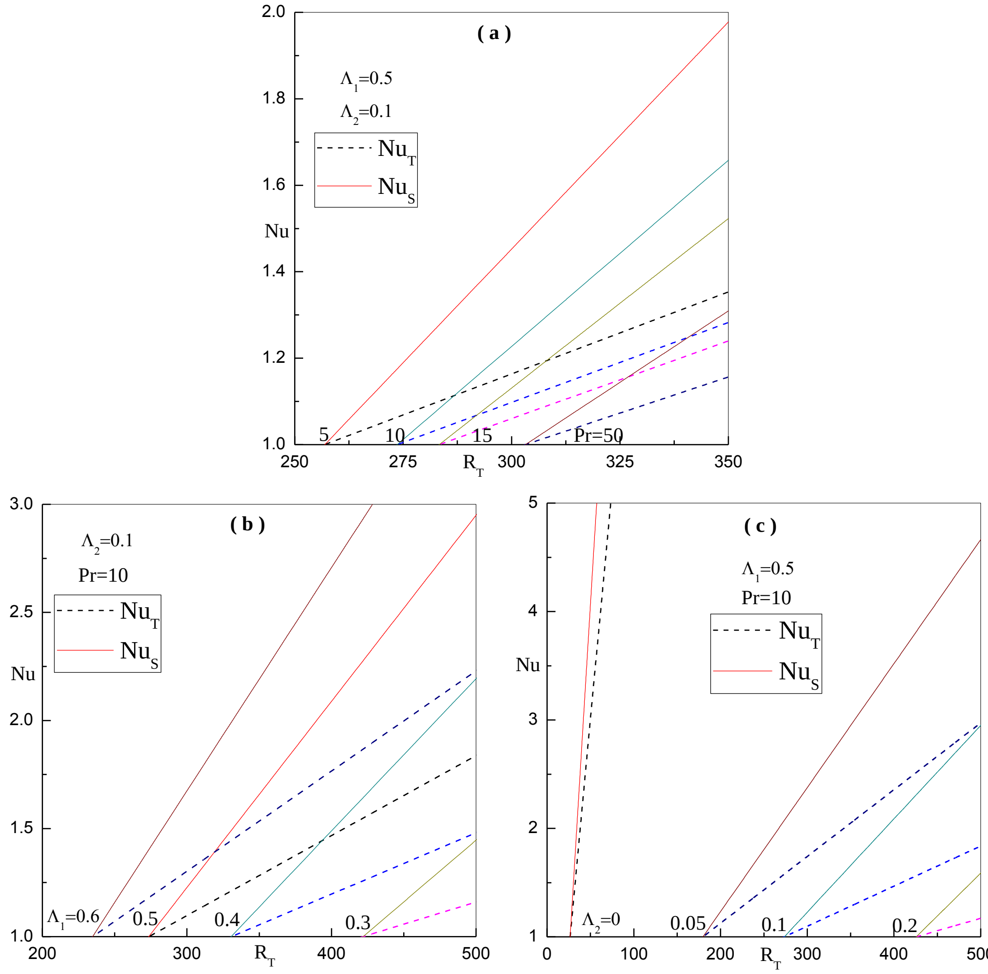

Fig. 6. Area-and time-averaged Nusselt numbers $\overline{Nu_T}$ and $\overline{Nu_S}$ for different values of (a) $\Pr$ (b) Λ_1 (c) Λ_2 when $\Gamma = 0.1$, $R_S = 100$.

The effects of elastic parameters on heat and mass transfer characteristics are summarized in Figs. 5(a, b) in terms of space averaged Nusselt numbers for stationary case. From these figures it is observed that the Nusselt numbers increase with increasing R_T / R_{Tc}. The effect of increasing Λ_1 (Fig. 5a) is to decrease the Nusselt number while increase in Λ_2 (Fig. 5b) exhibits an opposite kind of behavior. The curves of Nusselt numbers shown for equal values of

Λ_1 and Λ_2 correspond to the Newtonian case which lies well above their respective curves for $\Lambda_1 \neq \Lambda_2$. Thus heat and mass transfer is more vigor in Newtonian fluids compared to viscoelastic fluids. Figures 6(a-c) illustrate that heat and mass transfer rates increase with an increase in the value of relaxation parameter and a decrease in the retardation parameter as well as the Prandtl number. In either of the cases it is observed that $\overline{Nu_T} < \overline{Nu_S}$.

5. CONCLUSIONS

A weakly nonlinear stability of double-diffusive stationary and oscillatory convection in a viscoelastic fluid layer considering the Oldroyd-B constitutive equation is investigated. The cubic Landau equations are derived and the stability of stationary and oscillatory bifurcations in the neighborhood of their critical values is analyzed. The main observations are pointed out below.

- A codimension-two bifurcation occurs at well-defined parametric conditions and the value of Λ_1 at which it occurs decreases with increasing Prandtl number in the viscoelastic parameters plane.
- The viscoelastic parameters influence the stability of stationary bifurcation despite their effect is not felt on the stationary onset. The stationary bifurcation depends on viscoelastic parameters and both subcritical and supercritical bifurcations occur over a certain range of parameters. If the additional solute concentration field is destabilizing, then the stationary bifurcation always bifurcates supercritically.
- The oscillatory bifurcating solution becomes subcritical at higher values of solute Rayleigh number with increasing retardation parameter and Prandtl number, while an opposite trend could be seen with increasing relaxation parameter.
- Heat and mass transfer increases with increasing retardation parameter and decreasing relaxation parameter in the stationary case, while an opposite kind of behavior is noticed in the oscillatory case. The effect of increasing Prandtl number is to decrease the Nusselt numbers.

ACKNOWLEDGMENT

One of the authors (KRR) (SRF) wishes to thank the Department of Science and Technology, New Delhi for granting him a fellowship under the Innovation in Science Pursuit for the Inspired Research (INSPIRE) Program (No. DST/INSPIRE Fellowship/ [IF150253]) and also extends his gratitude towards the Principal and Management of his college for their encouragement and support. The authors wish to thank the reviewer for the useful suggestions.

REFERENCES

Altawallbeh, A. A., I. Hashim and B. S. Bhadauria (2017). On the linear stability analysis of double diffusive convection in a viscoelastic fluid saturated porous layer with cross diffusion effects and internal heat source. *AIP Conference Proceedings* 1830(1), 020008-020016.

Ashraf, M. B., T. Hayat and A. Alsaedi (2016). Radiative mixed convection flow of an Oldroyd-B fluid over an inclined stretching surface. *Journal of Applied Mechanics and Technical Physics 57*(2), 317-325.

Awad, F. G., P. Sibanda and S. S. Motsa (2010). On the linear stability analysis of a Maxwell fluid with double- diffusive convection. *Applied Mathematical Modelling 34*(11), 3509-3517.

Bird, R. B., R. C. Armstrong and O. Hassager (1987). Dynamics of polymeric liquids : Fluid mechanics.

Chen, C. F. and D. H. Johnson (1984). Double-diffusive convection: a report on an engineering foundation conference. *Journal of Fluid Mechanics 138*, 405-416.

Eltayeb, I. A. (1977). Nonlinear thermal convection in an elasticoviscous layer heated from below. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* 356(1685), 161-176.

Green, T. (1968). Oscillating convection in an elasticoviscous liquid. *The Physics of Fluids 11*(7), 1410-1412.

Hirata, S. C., L. D. B. Alves, N. Delenda and M. N. Ouarzazi (2015). Convective and absolute instabilities in Rayleigh–Bénard–Poiseuille mixed convection for viscoelastic fluids. *Journal of Fluid Mechanics 765*, 167-210.

Huppert, H. E. and J. S. Turner (1981). Double-diffusive convection. *Journal of Fluid Mechanics 106*, 299-329.

Khayat, R. E. (1995). Non-linear overstability in the thermal convection of viscoelastic fluids. *Journal of non-newtonian fluid mechanics 58*(2-3), 331-356.

Kolodner, P. (1998). Oscillatory convection in viscoelastic DNA suspensions. *Journal of non-newtonian fluid mechanics 75*(2), 167-192.

Kumar, A. and B. S. Bhadauria (2011). Non-linear two dimensional double diffusive convection in a rotating porous layer saturated by a viscoelastic fluid. *Transport in Porous Media 87*(1), 229-250.

Larson, R. G. (1992). Instabilities in viscoelastic flows. *Rheologica Acta 31*(3), 213-263.

Li, Z. and R. E. Khayat (2005). Finite-amplitude Rayleigh–Benard convection and pattern selection for viscoelastic fluids. *Journal of Fluid Mechanics 529*, 221-251.

Malashetty, M. S. and B. S. Biradar (2011). The onset of double diffusive convection in a binary Maxwell fluid saturated porous layer with cross-diffusion effects. *Physics of Fluids 23*(6), 064109-064113.

Malashetty, M. S. and M. Swamy (2010). The onset of double diffusive convection in a viscoelastic fluid layer. *Journal of Non-Newtonian Fluid Mechanics 165*(19), 1129-1138.

Malashetty, M. S., M. Swamy and R. Heera (2009). The onset of convection in a binary viscoelastic fluid saturated porous layer. *ZAMM- Journal of Applied Mathematics and Mechanics 89*(5), 356-369.

Martinez-Mardones, J. and C. Perez-Garcia (1990). Linear instability in viscoelastic fluid convection. *Journal of Physics: Condensed Matter 2*(5), 1281-1290.

Martinez-Mardones, J. and C. Perez-Garcia (1992). Bifurcation analysis and amplitude equations for viscoelastic convective fluids. *Il Nuovo Cimento D 14*(9), 961-975.

Martinez-Mardones, J., R. Tiemann and D. Walgraef (2000). Rayleigh–Bénard convection in binary viscoelastic fluid. *Physica A: Statistical Mechanics and its Applications 283*(1), 233-236.

Martínez-Mardones, J., R. Tiemann, D. Walgraef and W. Zeller (1996). Amplitude equations and pattern selection in viscoelastic convection. *Physical Review E 54*(2), 1478-1488.

Platten, J. K. and J. C. Legros (2011) *Convection in liquids*. Springer-Verlag, New York.

Raghunatha, K. R., I. S. Shivakumara and B. M. Shankar (2018) Weakly nonlinear stability analysis of triple diff usive convection in a Maxwell fluid saturated porous layer. *Applied Mathematics and Mechanics (English Edition)* 39, 153-168.

Rosenblat, S. (1986). Thermal convection in a viscoelastic liquid. *Journal of Non-Newtonian Fluid Mechanics 21*(2), 201-223.

Sokolov, M. and R. I. Tanner (1972). Convective stability of a general viscoelastic fluid heated from below. *The Physics of Fluids 15*(4), 534-539.

Swamy, M. S. and W. Sidram (2012). Effect of rotation on the onset of thermal convection in a viscoelastic fluid layer. *Fluid Dynamics Research 45*(1), 015504-015525.

Turner, J. S. (1974). Double-diffusive phenomena. *Annual Review of Fluid Mechanics 6*(1), 37-54.

Turner, J. S. (1985). Multicomponent convection. *Annual Review of Fluid Mechanics 17*(1), 11-44.

Turner, J. S. (1973). *Buoyancy effects in fluids*. Cambridge University Press, London.

Venezian, G. (1969). Effect of modulation on the onset of thermal convection. *Journal of Fluid Mechanics 35*(2), 243-254.

Veronis, G. (1968). Effect of a stabilizing gradient of solute on thermal convection. *Journal of Fluid Mechanics 34*(2), 315-336.

Vest, C. M. and V. S. Arpaci (1969). Overstability of a viscoelastic fluid layer heated from below. *Journal of fluid Mechanics 36*(3), 613-623.

Wang, S. and W. Tan (2008). Stability analysis of double-diffusive convection of Maxwell fluid in a porous medium heated from below. *Physics Letters A 372*(17), 3046-3050.

Wang, S. and W. Tan (2011). Stability analysis of soret-driven double-diffusive convection of Maxwell fluid in a porous medium. *International Journal of Heat and Fluid Flow 32*(1), 88-94.