Journal of Applied Fluid Mechanics, Vol. 12, No. 4, pp. 1223-1229, 2019.
Available online at www.jafmonline.net, ISSN 1735-3572, EISSN 1735-3645.
DOI: 10.29252/jafm.12.04.29610

Towards Methodologies for Optimal Fluid Networks Design

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(Received September 6, 2018; accepted November 3, 2018)

ABSTRACT

Tree flow networks are ubiquitous in nature and abound in engineered systems. A parent tube branching into two daughter tubes is the main building block of these networks. These branched tubes should be designed to provide easier access to flow under different size constraints. Optimal tree networks follow a homothetic scaling where the sizes of tubes have the same ratios between successive generations. In this study, different approaches aiming at optimal design of bifurcating tubes are presented and compared. The cross-sectional area of the tubes is obtained using two methods, based on Lagrange multipliers with a size constraint to respect, and including the size limitations directly into the function to optimize via chain rule. The optimal length of the tubes is obtained based both on the equipartition of forces/resistances and on the equal thermodynamic distance. These methods can be understood as a way of connecting entropy generation and the size of branching tubes. This study shows that applying the Lagrange Multiplier Method and applying the chain rule with constraint provides the same result. A similar result is obtained when the equipartition of forces/resistances and equal thermodynamic distance design methods are applied. These results are valid for different size constraints. In summary, our paper provides a comprehensive comparison of the different methods for a better choice, and is intended to provide insights into tree networks of tubes of any shape under different size constraints, for design and analysis.

Keywords: Tree-shaped flow networks; Optimal design; Lagrange multipliers; Chain rule; Equipartition of forces; Equal thermodynamic distance.

NOMENCLATURE

A	cross-sectional area
A_s	surface area
C_s	size constraint
D	diameter of tube
k	thermodynamic distance of each step
K	total thermodynamic
L	length of the tube
N	number of branching levels
p	pressure
Q	volumetric flow rate
R	flow resistance
S	entropy generation
V	volume
α	shape factor
λ	lagrange multiplier
μ	dynamic viscosity
θ	shape factor

Subscripts

d	daughter tubes
min	minimum
p	parent tube
j	tube at j^{th} branching level
0	tube at the initial branching level

1. INTRODUCTION

Tree-shaped flow networks are omnipresent in nature and play an important role in technology. These binary branching networks of tubes successively bifurcates up to a terminal level, and constitute the best access to connect a finite-size volume (or area) and one point (Bejan, 2000). This study presents and evaluates different methodologies for constraint-based design of tree-shaped flow networks by cost function optimization.

Optimal geometry generation obeys to a homothetic scaling. This mean that the sizes between successive generations of tubes have the same ratios. Since the pioneering works performed in the early 1900s (i.e., Hess, 1917 & Murray, 1926), several studies were devoted to the optimization of bifurcated tubes. Hess-Murray's law was derived for the vascular system based on principle of minimum work (Murray, 1926). It states that for a cylindrical parent blood vessel that branch into daughter vessels, the diameter ratio of these successive vessels are homothetic with $2^{-1/3}$.

By applying to the constructal law of design and using the resistance minimization, Bejan *et al.* (2000) confirmed the Hess-Murray law for laminar flow under volume constraint. The emergence of a tree-shaped network as the best architecture to connect the volume (area)-point flows can be also anticipated, based on the constructal law (Bejan, 2000).

Authors such as Horsfield and Cumming (1967), Kamiya and Togawa (1972), Zamir (1975), Bejan *et al.* (2000), Miguel (2015, 2019 and Miguel, 2019) have argued that Hess-Murray law is only valid for an isothermal laminar flow of Newtonian fluids in a tubes of rigid impervious walls. For this reason, several authors developed extensions to Hess-Murray law to account for turbulent flows, non-Newtonian fluids, flows in permeable tubes, etc. (see for example, Horsfield and Cumming (1967), Zamir (1975), Bejan *et al.* (2000), Miguel (2015)). Other authors (e.g., Bejan *et al.*, 2000; and Miguel, 2016a, 2016b) studied the functional relationship between tube lengths. In this context, they found that length ratio of successive tube segments obeys also to a homothetic scaling. A comprehensive review of homothetic scaling for optimal tree-shaped networks is given by Miguel and Rocha (2018).

The diameter ratio of successive tube segments was first derived from the principle of minimum work, and subsequently confirmed by minimizing the resistance, minimizing pumping power, minimizing entropy, etc. (Miguel and Rocha, 2018). These networks of tubes are also employed as flow distribution systems in designing heat sink (see for example Bahiraei *et al.*, 2017, 2018, 2019).

The optimal branching of fluidic networks fall under the category of constrained optimization. Because the space allocated to networks is a premium, size-limiting constraints such as fixed volume and surface area of tubes must be accounted (Gosselin, 2007). Another important feature is that complex flow systems are not only assemblies of circular tubes. Microfluidic networks are often made of non-circular tubes. To account for these shapes, cross-sectional area of tubes in finding the optimal geometry of bifurcated tube is desirable.

Here the design of a tree-shaped network under different size constraints is optimized, by means of different optimization methods. For the cross-sectional area of tubes, two methods are used, namely, the Lagrange Multiplier Method (as a constraint to respect), and applying the constraint directly into a global function to optimize, via chain rule. Two derived entropy optimization methods were applied for the optimal lengths of successive tube segments, namely, the equipartition of forces/resistances and the equal thermodynamic distance. In this study, a comparison between methods is provided. The performance of flow systems belongs to the type of constrained optimization problems, since networks are tailored to particular size-limiting requirements (volume and surface area). The goal is also to obtain analytical expressions for the optimum daughter–parent cross-sectional area and length ratios for branching networks of tubes of any cross-sectional shape under laminar flow.

2. Geometric Optimization Of Branched Tubes

Tree-shaped flow networks are complex branched distribution system. A dichotomous branching segment of tubes is the elementary building block of the network (Fig. 1). Consider a steady-state incompressible laminar flow of a Newtonian fluid with constant viscosity and density. The Hagen–Poiseuille flow resistance for a single tube is (Miguel, 2015)

$$R=\frac{\Delta p}{Q}=\frac{\alpha\mu L}{A^2} \tag{1}$$

where R is the flow resistance, p is the pressure, Q is the volumetric flow rate, L is the length of the tube, μ is the dynamic viscosity, A is the cross-sectional area of the tube and α is a shape factor (e.g., for cylindrical channels α=8π).

Fig. 1. Schematic representation of a binary-tree-shaped flow network. A parent tube branching into two daughter tubes is the main building block of the tree network of tubes.

Let the flow resistance at the junction of parent and daughter tubes be very small when compared with the flow resistance of parent and daughter tubes (i.e., the svelteness factor (Wechsatol *et al.*, 2006) defined by the ratio of the external to the internal length scales is higher than the square root of 10).

Let be the flow system composed by a parent tube that branches into two equal daughter tubes (i.e. symmetrical branching tubes). The resistances of daughter tubes are in parallel, and their overall resistance and the resistance of parent tube are in series. The total flow resistance of assembly of tubes is

$$R_{total}=R_p+0.5R_d=\alpha\mu\left(\frac{L_p}{A_p^2}+\frac{L_d}{2A_d^2}\right) \quad (2)$$

Here the subscripts p and d mean parent and daughter tubes, respectively. For obtaining the optimal design, constraints provide a way to account for a variety of size limitations. A fixed total volume and fixed total surface area are considered

$$V_{total}=V_p+V_d=A_pL_p+2A_dL_d \quad (3)$$

$$A_{s,total}=A_{s,p}+A_{s,d}=\theta A_p^{1/2}L_p+2\theta A_d^{1/2}L_d \quad (4)$$

where V is the volume, A_s is the surface area, and θ is a shape factor (e.g., for cylindrical channels $\theta=2\pi^{1/2}$).

The geometry of the assembly of tubes has the following degrees of freedom that are optimized: cross-sectional areas and lengths of the tubes. The Constructal Law is about the occurrence of design in flow systems and states that “for a finite-size flow system to persist in time it must evolve such that it provides greater and greater access to the currents that flow through it” (Bejan, 2000). Here we seek to minimize the total resistance subject to the volume and surface area constraints.

The Extreme Value Theorem states that a continuous function f on a bounded closed interval has absolute maximum and minimum values. The first derivative test for local extrema gives us a method to determine the value. If f is differentiable at an interior point c of its domain, then $f'(c)=0$. According to the second derivative test for concavity, if f is twice differential and $f''>0$ the graph of f is concave up (convex), then f has a minimum at c. Our goal is to find where the constrained total resistance has a minimum.

3. Optimal Cross-Sectional Areas of Bifurcation Segment

3.1 Constrained Optimization Using Lagrange Multipliers

The Lagrange multiplier method encompasses the modification of the objective function by adding terms that describe the constraints. The objective function is extended by constraint relations using non-negative multiplicative Lagrange multipliers. The resulting objective function is a function of the n_v design variables and n_λ Lagrange multipliers. To this end, we have to minimize the cost function $R_\lambda=R+\lambda C_s$, where λ is the Lagrange multiplier and C_s is the size constraint (volume, surface area). Note that R and C_s have well defined physical meaning and this assigns a physical meaning to the Lagrange multiplier.

Let be the volume the size-limiting constraint. Substitution of Eqs. (2) and (3) into the cost function R_λ yields

$$R_\lambda=\alpha\mu\left(\frac{L_p}{A_p^2}+\frac{L_d}{2A_d^2}\right)+\lambda\left(A_pL_p+2A_dL_d\right) \quad (5)$$

We seek to optimize the ratio A_d/A_p. According to the first derivative test for local extrema ($dR_\lambda/dA_p=0$ and $dR_\lambda/dA_d=0$) and the second derivative test for concavity ($d^2R_\lambda/dA_p^2>0$ and $d^2R_\lambda/dA_d^2>0$), Eq. (5) has a minimum valor at

$$\frac{A_d}{A_p}=2^{-2/3} \quad (6)$$

Equation (6) shows that the cross-sectional area ratio of successive tubes is homothetic with $2^{-2/3}$. This result agrees with the findings of Miguel (2018b). For cylindrical tubes with diameter D, Eq. (6) results in $D_d/D_p=2^{-1/3}$ which is the ratio between the diameters of daughter and parent tubes established by the Hess-Murray law.

Consider now an assembly of tubes with a surface area constraint. Substitution of Eqs. (2) and (4) into the cost function R_λ yields

$$R_\lambda=\alpha\mu\left(\frac{L_p}{A_p^2}+\frac{L_d}{2A_d^2}\right)+\lambda\left(\theta A_p^{1/2}L_p+2\theta A_d^{1/2}L_d\right) \quad (7)$$

The minimization of Eq. (7) yields

$$\frac{A_d}{A_p}=2^{-4/5} \quad (8)$$

This result is different than Eq. (6) for volume constrained systems. Note also that for cylindrical tubes with diameter D, Eq. (8) results in $D_d/D_p=2^{-2/5}$ which agrees with the results obtained by Gosselin (2007).

3.2 Constrained Optimization Using Chain Rule.

The composition or chain rule allows us to differentiating compositions of functions. Differentiation of total flow resistance (Eq. (2)) with respect to A_d, for example, results in

$$\frac{dR_{total}}{dA_d}=\frac{dR_{total}}{dA_p}\frac{dA_p}{dA_d}+\frac{dR_{total}}{dA_d} \quad (9)$$

The size-limiting constraints can be included into the first right-hand term of Eq. (9) by following Eqs. (3) and (4). For a fixed total volume

$$A_p=\frac{\left(V_{total}-2A_dL_d\right)}{L_p} \quad (10)$$

and for a fixed total surface area

$$A_p = \left(\frac{A_{s,total} - 2\theta A_d^{1/2} L_d}{\theta L_p} \right)^2 \tag{11}$$

In the said conditions above named, to optimize the ratio A_d/A_p, it is required that $dR_{total}/dA_d=0$ and $d^2R_{total}/dA_d^2>0$.

Let be a system with volume constraint. Substituting derivatives of Eqs. (2) and (10) into (9) yields

$$\frac{dR_{total}}{dA_d} = 4\alpha\mu \frac{L_p}{A_p^3}\frac{L_p}{L_d} - \alpha\mu \frac{L_d}{A_d^3} \tag{12}$$

and solving, it becomes

$$\frac{A_d}{A_p} = 2^{-2/3} \tag{13}$$

Consider now that the total surface area occupied by the tubes is fixed. Substituting derivatives of Eqs. (2) and (11) into (9), and the minimization of flow resistance yields

$$\frac{A_d}{A_p} = 2^{-4/5} \tag{14}$$

It is remarkable to note that Eqs. (6) and (13), and Eqs. (8) and (14) present the same result. This means that the constrained optimization using Lagrange multipliers and using composition or chain rule delivers the same optimization rules.

There is another method in which the constraint is inserted into the resistance to get a new (constrained) equation, as presented by Bejan *et al.* (2000). For a volume constrained in cylindrical bifurcated tubes, these authors found the optimal diameter ratio between parent and daughter tubes by solving the first derivative of the new equation. The well-known Hess-Murray law was obtained.

4. OPTIMAL LENGTHS OF BIFURCATION SEGMENT

The aim is now to obtain the optimal lengths of the assembly of tubes for both constrained total volume and constrained total surface area.

4.1 Design by Equipartition of Resistance Method

The principle of equipartition of entropy generation states that the generation of entropy is minimal when entropy production is uniformly distributed (Tondeur and Kvaalen, 1987; Bejan and Tondeur, 1998).

For isothermal flows through branching tubes, the entropy generation is (Bejan, 2006)

$$S_g = \frac{1}{T}\sum_i Q_i \Delta p_i \tag{15}$$

where S_g is the entropy generation. To hold continuity equation the total fluid flow rate Q at each branching level is a constant, and

$$Q = \frac{\Delta p_1}{R_1};\ Q = \frac{\Delta p_2}{R_2};\ldots\ Q = \frac{\Delta p_n}{R_n} \tag{16}$$

The minimum entropy generation (Eq. (15)) can be found by using the Lagrange method for all branching levels. The derivatives can be recognized from Eq. (16) and solving, results in

$$-Q = \lambda\frac{1}{R_1};\ -Q = \lambda\frac{1}{R_2};\ldots\ -Q = \lambda\frac{1}{R_n} \tag{17}$$

This means that

$$\frac{\lambda}{Q} = -R_1;\ \frac{\lambda}{Q} = -R_2;\ \ldots\frac{\lambda}{Q} = -R_n \tag{18}$$

and

$$\lambda = -\Delta p_1;\ \lambda = -\Delta p_2;\ \ldots\lambda = -\Delta p_n \tag{19}$$

Equations (18) and (19) show that the minimum entropy generation is obtained by equipartition of driving forces (pressures) or by equipartition of resistances

$$S_{g,min} = \frac{QN\Delta p_1}{T} = \frac{Q^2 N R_1}{T} \tag{20}$$

where N is the number of branching levels.

In summary, Eq. (20) states that the driving forces (pressures) or flow resistances should be a constant (in space and time) to obtain a minimum total entropy generation. In other words, $\Delta p_1=\Delta p_2=\ldots=\Delta p_n$ or $R_1=R_2=\ldots=R_n$.

In our study, the optimal design is obtained when $R_p=R_d$, i.e., flow resistances must be equally distributed (equipartitioned) along the system

$$\alpha\mu\frac{L_p}{A_p^2} = \alpha\mu\frac{L_d}{2A_d^2} \tag{21}$$

and

$$\frac{L_d}{L_p} = 2\left(\frac{A_d}{A_p}\right)^2 \tag{22}$$

According to Eq. (22), for optimal design of bifurcating tubes to emerge, it is necessary to include the optimal cross-sectional area ratio under a certain size constraint.

For a flow system with a fixed total volume, insertion of Eq. (6) into Eq. (22) gives

$$\frac{L_d}{L_p} = 2^{-1/3} \tag{23}$$

Equation (23) represents optimal lengths of the assembly of tubes with volume constraint, and agrees with the result obtained by Bejan *et al.* (2000), where the size limitations are introduced directly into the cost function to be minimized.

For a system with surface area constraint, the combination of Eqs. (8) and (22) results in

$$\frac{L_d}{L_p} = 2^{-3/5} \tag{24}$$

This equation represents the optimal daughter–parent diameter ratio for a system with fixed surface area.

4.2 Design by Equal Thermodynamic Distance

The equal thermodynamic distance principle states that a system, which is allowed to exchange fluid and heat with its surroundings, must go from one thermodynamic state to another in equally long steps in order to minimize the total generation of entropy (Salamon and Berry, 1983; Nulton *et al.*, 1985; Andresen and Gordon, 1994).

As we know, entropy is generated when fluid flows through the tubes. The minimum entropy generation during the process is (Nulton *et al.*, 1985; Andresen and Gordon, 1994)

$$S_{min} = \frac{K^2}{2N} \tag{25}$$

where S_{min} is the minimum entropy production, and K is total thermodynamic distance from inlet to outlet given by

$$K = \sum_{i=1}^{N} k_i \tag{26}$$

Here k is the thermodynamic distance of each step (e.g., at each tube branching). Nulton *et al.* (1985) pointed out that this equality is obtained only when all k are equal. In other words, the thermodynamic distance of each step must be equally distributed along the system.

Different extensive and intensive properties may be involved in the process. In our study, the focus is an isothermal flow through bifurcating tubes, and $k=(Q\Delta p/T)^{1/2}$.

The optimal design is obtained when parent and daughter tubes are equally long in the thermodynamic state space ($k_p=k_d$), to minimize the total entropy production. As the volumetric flow rate at each branching level is a constant

$$\left(\frac{\alpha\mu Q^2 L_p}{TA_p^2}\right)^{1/2} = \left(\frac{\alpha\mu Q^2 L_d}{2TA_d^2}\right)^{1/2} \tag{27}$$

and

$$\frac{L_d}{L_p} = 2\left(\frac{A_d}{A_p}\right)^2 \tag{28}$$

It is important to note that Eq. (28) is equal to Eq. (22). This means that, although having different starting points, both methods of equipartition of forces or resistances and equal thermodynamic distance aim to minimize the total entropy generation in the system, by distributing it evenly among the bifurcating levels and producing the same result.

Substituting Eqs. (6) and (8) into Eq. (28), Eqs. (23) and (24) are recovered, as expected.

5. Homothetic Relations For Optimal Design Of A Tree-Shaped Flow Network

According to Eqs. (6), (8), (23) and (24), the optimal size ratio between tube segments follows the homothetic relations

$$\frac{A_d}{A_p} = \omega_A \tag{29}$$

$$\frac{L_d}{L_p} = \omega_L \tag{30}$$

where scale factors ω_A and ω_L are constants. For volume-limiting and surface area-limiting constraints are $\omega_A=2^{-2/3}$ and $\omega_L=2^{-1/3}$, and $\omega_A=2^{-4/5}$ and $\omega_L=2^{-3/5}$, respectively.

Consider a tree-shaped network (Fig. 1) formed by j branches of tubes between level 0 to N. Homothetic scalings can be expressed in terms of size of tube at the initial branching level and at j-branching level

$$\frac{A_j}{A_0} = \omega_A^j \tag{31}$$

$$\frac{L_j}{L_0} = \omega_L^j \tag{32}$$

where the subscripts 0 and j mean tubes at level 0 (i.e., the tube at the initial branching level) and at level j, respectively.

6. Conclusions

An important step in the optimization process is the choice of the methodology. In this study, a comparison between methods is provided. By means of the total resistance minimization argument, the cross-sectional area ratio was obtained applying both the Lagrange multiplier that is used in constrained variational problems (as a constraint to respect), and including the size-constraint requirement directly in function to optimize via chain rule. The homothetic scaling result obtained with both methods is the same. This shows that both methods are equivalent.

Using thermodynamics arguments (i.e., minimization of entropy generation), the equipartition of forces/resistances and equal thermodynamic distance design methods were applied to obtain the length ratio of successive tubes segments. Although different methods, it was found that the homothetic ratio for lengths obtained were the same, indicating that the methods are equivalent. While these methodologies require the homothetic ratio for the cross-sectional areas, they are straightforward to use and to interpret, constituting excellent alternatives to obtain the length ratios of successive tubes.

It is also important to note that the size-limiting constraints affects the homothetic ratios for cross-sectional area and length of successive tubes. Both homothetic ratios for cross-sectional area and length are larger when the volume is fixed than when the surface are is constrained.

Another important conclusion is that the resistance of an optimal tree-shaped flow formed by N branches can be simply written as

$$R = NR_0 = N\frac{\alpha\mu L_0}{A_0^2}$$

where R_0 is the flow resistance of tube at the initial branching level.

In summary, here some guidance to help with the choices of optimization methods is provided. The results obtained may also contribute to a deeper understanding of tree networks indispensable for the design of flow systems with tubes of any shape. In a future study, constraint-based design under non-isothermal conditions and asymmetric branching should be addressed.

ACKNOWLEDGEMENTS

This research was supported by the project ALT20-03-0145-FEDER-029624.

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